MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1933 MARCH 10

Professor F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

C. J. M. Aanensen, 165 Moorland Road, Splott, Cardiff ;
S. Chandrasekhar, Trinity College, Cambridge ;
Gabriel Gideon Cillié, Hertford College, Oxford ;
Raymond Edward Victor Ely, Salcombe Lodge, Ewell, Surrey ;
James de Graaff Hunter, M.A., Sc.D. (late Director of the Geodetic Branch, Survey of India), 9 Westminster Park, Chester ;
Rev. Franklin Isaac Hutchinson, D.Litt., B.Ph., The Rectory, Barningham with Coney Weston, near Bury St. Edmunds ;
Harald von Klüber, Dr.Phil., Einstein-Institut, Potsdam, Germany ;
Percy George Murden, 267 High Holborn, London, W.C. 1 ; and
Harry B. Rumrill, Conestoga Road, Berwyn, Pennsylvania, U.S.A.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Vladmir Guth, Ph.D., Assistant at the National Observatory, Prague, 11 Preslova, Praha xvi–Smichov, Czechoslovakia (proposed by H. Slouka);
S. Kawasaki, Mizusawa Observatory, Japan, and 247 Hills Road, Cambridge (proposed by H. Jeffreys) ;
Miss Bohumila Nováková, Ph.D., 4 Velehradská, Praha xii, Czechoslovakia (proposed by H. Slouka) ;

R. W. B. Pearse, A.R.C.S., D.I.C., Ph.D., Assistant Lecturer in Astrophysics, Royal College of Science, S. Kensington, London, S.W. 7 (proposed by A. Fowler); and

R. H. Stoy, B.A., Gonville and Caius College, Cambridge (proposed by R. O. Redman).

One hundred and eight presents were announced as having been received since the last ordinary meeting, including :—

J. Jackson, "The Sun at Greenwich Apparent Noon, 1775–1798";

J. Jackson, "The Sun at Oxford Apparent Noon, 1775–1798";

E. A. Milne, "World-structure and the Expansion of the Universe";

Paris Academy of Sciences, "Les Membres et les Correspondants de l'Académie Royal des Sciences, 1666–1793."

The President announced that Dr. V. M. Slipher would deliver the George Darwin Lecture for 1933 at the Ordinary Meeting of the Society on May 12, taking as his subject "Spectrographic Studies of the Planets."

UPPER LIMITS TO THE DENSITY AND TEMPERATURE IN A STAR. (Second Paper.)

Sir A. S. Eddington, F.R.S.

1. A method of calculating upper limits to the density and temperature in a star of known mass was published by the author in 1931 ; * but the results there derived were not corrected for relativistic degeneracy, which makes an important difference. E. C. Stoner † has since introduced the relativity correction. Whilst his paper supplies the amended numerical results, there is still need for an appropriate restatement of the original proof, for the interest of these upper limits depends on the wide generality and rigour of the argument. In restating it here I have been able to remove some of the original restrictions and to simplify the argument.

Let ρ_0 be the greatest density inside a given star of mass M. Presumably ρ_0 is the central density, but we shall not assume this. We first prove that the pressure cannot anywhere exceed

$$P_{\max} \equiv B\rho_0^{\frac{4}{3}}, \tag{1}$$

where $B = \frac{1}{2}(\frac{4}{3}\pi)^{\frac{1}{3}}GM^{\frac{2}{3}}$, G being the constant of gravitation. The greatest pressure is necessarily at the centre, and its value is

$$P_c = \int_0^R g\rho dr = \frac{G}{4\pi}\int_0^M \frac{M_r dM_r}{r^4} = \frac{G}{8\pi}\int_0^{M^2} \frac{d(M_r^2)}{r^4}, \tag{2}$$

M_r being the mass within a radius r.

Thus P_c is greatest if the successive elements $d(M_r^2)$ are all as close to

* *M.N.*, **91**, 444, 1931. † *Ibid.*, **92**, 662, 1932.

the centre as possible (having regard to the datum that the greatest density is ρ_0), *i.e.* when the matter is arranged as a sphere of uniform density ρ_0. This gives the limit P_{max} above stated.

In material of specified chemical constitution (in thermodynamic equilibrium) the material pressure p_G is determined by an equation of state $p_G = f(\rho, T)$. Keeping ρ fixed, p_G is a minimum at some particular temperature, usually at $T = 0$. Let the minimum be $F(\rho)$. Practically $F(\rho) = f(\rho, 0)$; but I define it as above in order to include substances which behave exceptionally, *e.g.* water near 4° C.

The calculation of $F(\rho)$ is purely a question of theoretical physics. I shall assume that existing theory provides sufficiently trustworthy values

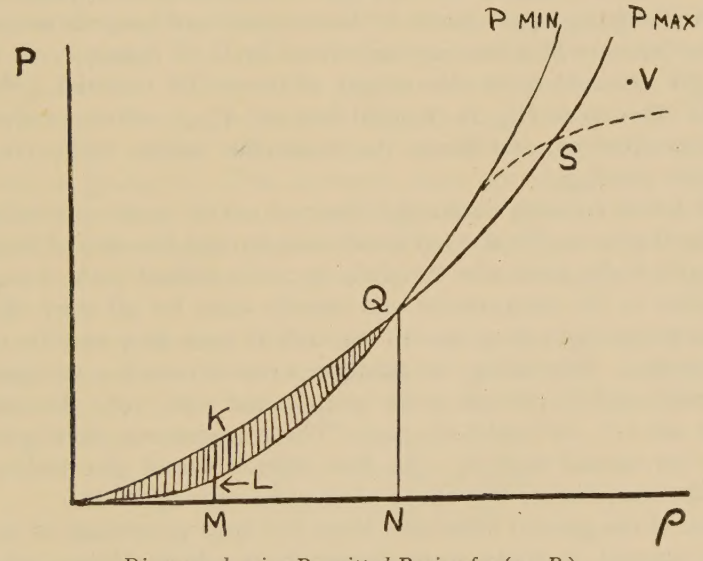

Diagram showing Permitted Region for (ρ_0, P_0).

of $F(\rho)$, or at least of a lower limit to $F(\rho)$, over the range of ρ which will be required. Then if P_0 is the pressure in the star at the place where the density is ρ_0, we have a lower limit to $(p_G)_0$ and *a fortiori* to P_0, viz.

$$P_{min} = F(\rho_0). \qquad (3)$$

2. With P as ordinate and ρ_0 as abscissa plot the curves (1) and (3). Suppose that they intersect at Q, as shown in the diagram. Then the point (ρ_0, P_0) corresponding to the actual values in the star must lie below P_{max} and above P_{min}, so that it must be in the shaded area between the two curves. It follows that Q gives the upper limit to the density and the upper limit to the pressure that can occur in a star of mass M. Since P includes the radiation-pressure $\frac{1}{3}aT^4$, we have also an upper limit to the temperature, viz. $\frac{1}{3}aT^4 < QN$.

These upper limits to the pressure and temperature apply to the whole star even when ρ_0 is not at the centre, for the curve P_{max} gives the greatest possible pressure *anywhere* in a star whose maximum density is ρ_0.

A more stringent upper limit to the temperature is obtained if we concede that the greatest density ρ_0 is at the centre.* Considering an ordinate KLM, the minimum pressure LM includes only the material pressure, so that the radiation-pressure $\frac{1}{3}aT^4$ must be included in KL. Hence, selecting the place where the vertical width of the shaded area is greatest, we have the upper limit, $\frac{1}{3}aT^4 < KL$.

The upper limits are independent of all astronomical hypotheses. They depend only on the validity of the current formulæ of atomic physics giving the equation of state of matter, and hence $F(\rho)$. I do not assert that these physical formulæ are necessarily final; but inasmuch as they are adopted by all recent writers, and numerous papers have been published dealing with their possible consequences, it is important to recognise that they automatically impose the foregoing upper limits to the densities and temperatures of stars.

3. The function $F(\rho)$ has been calculated by E. C. Stoner.† It depends only on the assumed molecular weight of the stellar material. When we change M, the curve $P_{\max}$ is changed but not $P_{\min}$; when we change the chemical constitution, and hence the molecular weight, the curve $P_{\min}$ is changed but not $P_{\max}$.

Upper limits are only obtained if the two curves cross. It is found by Stoner that they cross for stars of small mass but not for stars of large mass. For example, if the molecular weight μ is 2, the critical mass is $0{\cdot}436 \times \odot$. Upper limits to the temperature and density exist for all stars of smaller mass, though for stars very near to the critical mass they may be too high to be of interest. But taking, for example, a star of mass 0·3, the upper limit to the density is $8{\cdot}7 \cdot 10^6$ and to the temperature $1{\cdot}76 \cdot 10^9$. For mass 0·2, the limits are $1{\cdot}7 \cdot 10^6$ and $1{\cdot}07 \cdot 10^9$. The temperatures here given were found by my second method; the first method would give rather higher limits to the temperature but of the same order of magnitude.

In view of the general belief that there is a large percentage of hydrogen in stellar material, it might seem reasonable to adopt a lower value of μ, say $\mu = 1$. The critical mass is then $1{\cdot}74 \times \odot$; and for the Sun the upper limits are $\rho = 1{\cdot}8 \cdot 10^6$, $T = 1{\cdot}4 \cdot 10^9$. But there is some risk in assuming that the hydrogen extends actually to the centre of the star; and the assumption would in any case conflict with our aim, viz. to obtain conclusions entirely independent of astronomical hypotheses.

It may perhaps be objected that if we suppose the material near the centre to consist of the heaviest elements and so adopt the maximum molecular weight about 2·5, we just fail to obtain upper limits in the stars of smallest mass certainly known; so that our result might have no actual application. It is, however, to be remembered that the upper limits here given are very generous; and the objection only means that if every possibility is strained against us the generosity is just a little greater than we can afford.

The main object in establishing an upper limit of about 10^9 degrees for the temperature of a star is to show that excessively high temperature is not

* This second method gives an upper limit to the temperature at ϱ_0, so that if ϱ_0 is not at the centre the central temperature can be higher.

† *Loc. cit.* The numerical results in this section are quoted from his paper.

a necessary condition for the abundant liberation of subatomic energy. For this purpose we need only consider the lowest stellar mass that is reasonably abundant.

4. I wish now to show that these upper limits are not affected by any possible behaviour of stellar matter under pressures greater than QN. Ultimately we must come to pressures beyond the range of Stoner's calculations, with which physical theory is as yet unable to deal. Thus we cannot disprove the possibility that $P_{\min}$ follows the dotted line and intersects $P_{\max}$ again at S—though there is no reason to believe that it does.

There is then another region in which (ρ_0, P_0) can lie so as to be below $P_{\max}$ and above $P_{\min}$, *e.g.* at V. We must show that the state represented by V is impossible, otherwise our upper limits will have failed.

There is no way by which a star of mass M can reach V, for at some stage of development its maximum density ρ_0 would have to be intermediate between Q and S. The intrinsic pressure in the matter ($> P_{\min}$) is then greater than the maximum pressure ($P_{\max}$), which the material of the star can exert. The matter will therefore immediately expand, and the star will travel to Q, not S. This, however, does not completely disprove the possibility of the state V, for the star might have condensed as a large mass (in which there is no upper limit to the density) and have afterwards reduced its mass to M by radiation or ejection. The following proof is therefore required.

Let ρ_1 be any density in the star less than the maximum density ρ_0. Proceeding inwards from the surface, let the density first reach ρ_1 at radius r_1, so that outside r_1 we have $\rho < \rho_1$. Let M_1 be the mass interior to r_1, and let P_1 be the pressure at r_1. Then, as in (2),

$$P_1 = \int_{r_1}^{R} g\rho dr = \frac{G}{8\pi}\int_{M_1^2}^{M^2} \frac{d(M_r)^2}{r^4}. \qquad (4)$$

We see from (4) that P_1 is increased at each stage (*a*) by arranging the matter outside r_1 as a shell of uniform density ρ_1; (*b*) by adding matter on the outside so as to extend the uniform shell until its total mass is M; (*c*) by then reducing r_1 to zero so that P_1 becomes the pressure at the centre of a sphere of mass M and uniform density ρ_1. The final value of P_1 is accordingly $B\rho_1^{\frac{4}{3}}$. Thus for the original P_1 we have

$$P_1 < B\rho_1^{\frac{4}{3}}, \qquad P_1 > F(\rho_1). \qquad (5)$$

Hence (ρ_1, P_1) is subject to the same conditions and restricted to the same regions in the diagram as (ρ_0, P_0).

Thus not only (ρ_0, P_0) but a continuous line joining it to the origin (giving (ρ_1, P_1) at all points between the surface and the place of maximum density) must lie in the permitted regions of the diagram. Hence (ρ_0, P_0) cannot be in a region disconnected from the origin, and the state V is accordingly ruled out.

The possibility of a discontinuity of density in the star requires special consideration; it would involve a horizontal break in the line above-mentioned which joins (ρ_0, P_0) to the origin. The pressure in the star is necessarily

continuous, so that the break must be strictly horizontal. The discontinuity thus permits a horizontal jump from the shaded area to another permitted region. But this requires that matter can exist in two phases (one of them having exceedingly high density) *below* the pressure corresponding to Q.

Thus we are in no way concerned with the possibility of matter "solidifying" with a discontinuous change of density under pressures greater than those yet investigated theoretically. The only escape from our upper limits is to assume that we are insufficiently acquainted with the possible states of matter under ordinary white dwarf-pressure. Present theories of atomic physics give no hint of a second state of matter at such pressures; it would presumably involve a breakdown of the exclusion principle. As already explained, our single assumption is that existing theoretical physics is trustworthy up to white dwarf-pressures; so that we have guarded our theorem against this failure.

5. These upper limits are occasioned by the imperfection (degeneracy) of matter; for matter which is a perfect gas under all conditions $F(\rho)$ is zero, so that no upper limit is found. Hypotheses as to the existence of dense central cores in stars have generally been associated with the theoretical investigation of degenerate matter; but it should be understood that degeneracy does not help to make these high densities possible. It is an *adverse* factor. The common-sense view holds good—that it is easier to obtain high density with perfect gas than with relatively incompressible material.

In the more massive stars the temperature may rise so steeply (in comparison with the density) that the material never approaches degeneracy.* There is then no important deviation from a perfect gas. Since imperfection of the material is no longer a factor in the problem, the foregoing method can give no upper limits. That is the reason why the method only succeeds up to a critical mass.

When the present method gives no upper limits, we must not jump to the conclusion that very high temperatures and densities are possible. That depends on other properties of the material besides its equation of state—notably on the law of opacity. But since the physical theory of opacity is less fundamental than the theory of the equation of state and is not regarded with the same confidence, there is not the same interest in attempting to fix rigid upper limits on that basis. It is more appropriate to follow the usual methods which attempt to determine the actual central density and temperature so far as our present resources permit.

* *Cf.* S. Chandrasekhar, *Zeits. f. Astrophysik*, 5, 323, 1932.

THE MASS-PARTICLE IN AN EXPANDING UNIVERSE.

G. C. McVittie, Ph.D.

I. *Introduction and Summary*

In the astronomical applications of General Relativity two types of metric for the universe are used. For discussing the motion of planets round the Sun, the statical Schwarzschild metric is employed, which may be written in isotropic co-ordinates * as

$$ds^2 = \left(\frac{1 - m/2r_1}{1 + m/2r_1}\right)^2 dt^2 - \frac{1}{c^2}\left(1 + \frac{m}{2r_1}\right)^4 \{dr_1^2 + r_1^2(d\theta^2 + \sin^2\theta d\phi^2)\}; \qquad (1)$$

on the other hand, for dealing with the phenomenon of the recession of the spiral nebulæ non-statical metrics are used, which can be subdivided into two classes : the Lemaître class, in which

$$ds^2 = dt^2 - \frac{e^{\beta(t)}}{c^2}\left\{\frac{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)}{(1 + r^2/4R^2)^2}\right\}, \qquad (2)$$

and the de Sitter class, in which

$$ds^2 = dt^2 - \frac{e^{\beta(t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \qquad (3)$$

In (2) the constant $1/R^2$, which may be positive or negative, gives the curvature of space as a whole, local irregularities being disregarded. In (3) the curvature of space is zero. A particular case of (3) is the original de Sitter universe, for which $\beta(t) = t/a$ where a is a constant.†

One important respect in which the metric (1) differs from (2) and (3) is that in the former the co-ordinate r_1 is what we shall call an "observer's" co-ordinate, *i.e.* it is one based on the assumption that the distance between two points at relative rest is independent of the time. In (2) and (3) the co-ordinate r is one which will be called "cosmical." It is used when the system of nebulæ is taken as the basis of reference. An observer in the field uses a co-ordinate $r_1 = re^{\frac{\beta(t_1)}{2}}$ at the instant t_1; so that the "observer's" co-ordinate for a fixed value of r is not independent of the time.

When we consider that we are compelled to observe the universe from the near neighbourhood of a mass-particle (the Sun) it becomes of some interest to find a form of metric which will reduce to the statical metric (1) in terms of observer's co-ordinates, but which can also be expressed, approximately at least, in one of the forms (2) and (3) when cosmical co-ordinates are used. Some solutions of the problem have already been proposed. G. Lemaître ‡

* A. S. Eddington, *Math. Th. of Relativity*, 1924, § 43.

† H. P. Robertson, *Nat. Acad. Sci. Proc.*, **15**, 822–829, 1929.

‡ G. Lemaître, *M.N.*, **91**, 490–501, 1931.

has put forward one type of metric. Unfortunately his solution appears to depend on the assumption that the pressure of the matter outside the mass-particle is negative if the density is positive or *vice versa*. This condition is true whatever the co-ordinate system used. It is difficult to see how such a case could be applied to the actual universe unless, indeed, both pressure and density were zero. But this reduces Lemaître's result to the well-known one of a mass-particle in an, otherwise empty, de Sitter universe.* An alternative type of metric put forward by Dr. W. H. McCrea and the author † is also open to criticism on the ground that it implies that the matter in the universe outside the Sun is flowing toward it with a high velocity which is certainly not observed.‡

Hitherto it has been assumed that the problem can be solved by the choice of any set of "polar" co-ordinates, with origin at the mass-particle, that happen to yield some mathematical simplification of the equations involved. It is therefore hardly surprising that the resulting metrics do not apply to the actual universe. In the present investigation we have started from the point of view that an observer is necessarily constrained, by the methods he uses for setting up a co-ordinate system, to those systems in which the metric of space-time is expressed in an isotropic and orthogonal form. The predictions of any theory must, therefore, be calculated with respect to such a system before they can be compared with the observer's measurements.

We shall suppose that our observer takes into account the gravitational effect of the matter in the universe, but outside the mass-particle, by assuming that it is evenly spread out through space as if it were a gas. It will thus be characterised by (1) its density, T_4^4; (2) its pressure-components, T_1^1, T_2^2, T_3^3; and (3) its momentum, T_{14}, if it is flowing towards or away from the mass-particle. These are the only non-zero components of the energy-tensor when spherical symmetry is assumed. The pressure is, of course, due to the random velocities of the particles constituting the gas. Both density and pressure may, however, be in part due to the presence of radiation. It appears to us axiomatic that, in any space-time model applicable to the actual universe, the density and pressure cannot be negative but may, of course, be zero in a first approximation.

The solution of the problem now follows if the observer makes the general assumption that the mass-particle does not occupy a peculiar point in the distribution of matter in the universe. This leads him to conclude that, firstly, the pressure is everywhere isotropic, and secondly, that the matter is "at rest" with respect to his co-ordinate system. By this is meant that it has, on the whole, zero co-ordinate velocity, and therefore zero momentum, in his system. It might be thought that, in the actual universe, this could not be true because of the phenomenon of the "recession" of the nebulæ. But it must be emphasized that we do not *observe* a velocity in this connection,

* A. S. Eddington, *loc. cit.*, § 45.

† W. H. McCrea and G. C. McVittie, *M.N.*, **91**, pp. 128–133, 1930; *ibid.*, **92**, 7–12, 1931.

‡ G. C. McVittie, *M.N.*, **92**, 512 *et seq.*, 1932.

but only a shift to the red of the lines of the spectrum in the light emitted by distant objects. It is the whole object of the "expanding universe" theory to show that *even if* an observer assigns zero velocity to the nebulæ at each instant, yet this red-shift will be observed owing to the properties of space.

In arriving at the generalized Schwarzschild field we do not employ observer's co-ordinates, in the first instance. Instead we obtain the metric, using cosmical co-ordinates analogous to those used in (2) and (3). We then show that the metric can be transformed into the statical field (1) independently of the moment in time at which the transformation is made. This possibility turns out to be closely bound up with the condition that the matter in the universe has zero velocity, on the average, in the observer's system. If the mass-particle has a planetary system we show that it appears to be permanent and unchanging to the observer. The distant nebulæ, meanwhile, exhibit the recession phenomenon. We also point out that the use of cosmical co-ordinates, for dealing with this phenomenon, in no way implies that there is a "cosmical" time common to all observers. Lastly, we give reasons for believing that the observer must conclude that the cosmical constant λ cannot be zero or negative, and we discuss the present position of the theory of the disturbance of the equilibrium of the Einstein universe by the formation of mass-particles.

I wish to express my thanks to Sir Arthur S. Eddington for some helpful criticisms and suggestions.

II. *Equations Determining the Metric of Space-time*

We consider an observer engaged in setting up a co-ordinate system in the neighbourhood of a mass-particle (*e.g.* the Sun), which he takes as his origin of spatial co-ordinates. He is provided with rigid measuring rods and makes use of light-triangulations for dealing with points he cannot reach with his measuring rods. He works under the assumptions made by terrestrial observers, viz. :

(*a*) The length of a measuring rod is constant in time and independent of orientation around a given point.

(*b*) The backwards and forwards velocity of light between any two points is the same.

(*c*) The velocity of light is the same in every direction around a given point.

Under these circumstances he sets up an "observer's" co-ordinate system of the orthogonal and isotropic type,* in terms of which he expresses the metric applicable to the whole universe. In order to determine the coefficients of this metric he will have to make some assumptions regarding the distribution of matter in the universe. If he believes that his part of the universe is similar to every other part (except for the "singularity" corresponding to the mass-particle), he will be entitled to assume the following :—

(*d*) The matter in the universe is distributed with spherical symmetry around the origin where there is a mass-particle.

* A. S. Eddington, *loc. cit.*, § 43.

(*e*) There is no flow of the matter as a whole either towards or away from the origin—otherwise it would be necessary to postulate that, at some time or other, the neighbourhood of the origin had been the scene of an explosion (for instance) great enough to set the matter in the whole universe in motion away from that point. Until some physical process, capable of producing an upheaval of such magnitude, is discovered, our observer will be constrained to postulate (*e*).

(*f*) At any point in the universe the pressure is isotropic. This seems a natural consequence of (*e*) since there is now no preferential direction towards which the velocities of the particles, or the flow of radiation, might be directed.

Our object is now to find, by means of Einstein's equations, the metric which the observer assigns to the universe in this way. We shall not, however, determine it in terms of observer's co-ordinates directly, but instead find a metric which satisfies the requirements (*d*) to (*f*) in terms of isotropic cosmical co-ordinates. We shall then show that, on transforming this metric into observer's co-ordinates, the properties (*a*) to (*f*) are all found to hold. We deduce that this metric is the one actually used by our observer.

Consider, then, the most general form of metric which is orthogonal, isotropic in the space co-ordinates and which expresses the condition for spherical symmetry around the origin. Using cosmical co-ordinates, it can be written as

$$ds^2 = e^{\zeta(r,t)}dt^2 - \frac{e^{\nu(r,t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{4}$$

The distribution of density and pressure is given by Einstein's equations

$$G_{ik} - \lambda g_{ik} = -\kappa c(T_{ik} - \tfrac{1}{2}g_{ik}T), \tag{5}$$

where G_{ik} is the Riemann-Christoffel tensor; λ the cosmical constant; T_{ik} the energy-tensor; T the proper density, and κ is a constant related to the gravitational constant, γ, and the velocity of light, c, by

$$\frac{\kappa}{c} = \frac{8\pi\gamma}{c^4} = 2 \times 10^{-48} \text{ c.g.s units.}$$

We write

$$T_4{}^4 = \rho\,; \qquad T_1{}^1 = -p_1\,; \qquad T_2{}^2 = T_3{}^3 = -p_2\,;$$

so that

$$T = \rho - p_1 - 2p_2.$$

The equations (5) reduce to

$$-\frac{\kappa c}{2}(\rho + p_1 + 2p_2) + \lambda = e^{-\zeta}\{\tfrac{3}{2}\ddot{\nu} + \tfrac{3}{4}(\dot{\nu})^2 - \tfrac{3}{4}\dot{\zeta}\dot{\nu}\} - c^2e^{-\nu}\left\{\tfrac{1}{2}\zeta'' + \tfrac{1}{4}(\zeta')^2 + \frac{1}{r}\zeta' + \tfrac{1}{4}\zeta'\nu'\right\}, \tag{6}$$

$$-\frac{\kappa c}{2}(2p_2 - p_1 - \rho) + \lambda = e^{-\zeta}\{\tfrac{1}{2}\ddot{\nu} + \tfrac{3}{4}(\dot{\nu})^2 - \tfrac{1}{4}\dot{\nu}\dot{\zeta}\} - c^2e^{-\nu}\left\{\tfrac{1}{2}\zeta'' + \tfrac{1}{4}(\zeta')^2 - \tfrac{1}{4}\zeta'\nu' + \nu'' + \frac{1}{r}\nu'\right\}, \tag{7}$$

$$-\frac{\kappa c}{2}(p_1-\rho)+\lambda=e^{-\zeta}\{\tfrac{1}{2}\ddot{\nu}+\tfrac{3}{4}(\dot{\nu})^2-\tfrac{1}{4}\dot{\nu}\dot{\zeta}\}$$
$$-c^2e^{-\nu}\left\{\tfrac{1}{2}\nu''+\frac{3}{2r}\nu'+\tfrac{1}{4}(\nu')^2+\frac{1}{2r}\zeta'+\tfrac{1}{4}\nu'\zeta'\right\}, \quad (8)$$

$$-\kappa cT_{14}=\dot{\nu}'-\tfrac{1}{2}\dot{\nu}\zeta', \quad (9)$$

where a dot denotes differentiation with respect to t and a dash differentiation with respect to r.

The two conditions (e) and (f) are now expressed by

$$T_{14}=0 \quad \text{and} \quad p_1=p_2.$$

These lead, together with (7) and (8), to our two fundamental equations for determining the coefficients of the metric (4), viz. :

$$\dot{\nu}'-\tfrac{1}{2}\dot{\nu}\zeta'=0, \quad (10)$$

$$\zeta''+\nu''-\frac{1}{r}(\nu'+\zeta')-\nu'\zeta'-\tfrac{1}{2}(\nu')^2+\tfrac{1}{2}(\zeta')^2=0. \quad (11)$$

III. *Solution of the Equations*

We shall now show that ν and ζ can be determined by the use of (10) and (11) alone.

The equation (10) can be solved immediately. Dividing throughout by $\dot{\nu}$, we can write it as

$$\frac{\partial(\log\dot{\nu})}{\partial r}=\frac{1}{2}\frac{\partial\zeta}{\partial r},$$

whence

$$\dot{\nu}=\alpha(t)e^{\frac{1}{2}\zeta(r,\,t)}.$$

Hence finally

$$\nu=\int\alpha(t)e^{\frac{1}{2}\zeta(r,\,t)}dt+a(r), \quad (12)$$

where, in the integral, r is treated as a constant and $a(r)$ is a function of r alone.

We shall now show that the generalized Schwarzschild field we are seeking cannot be included amongst those solutions for which ζ is a function of r alone. For, in such a case,

$$\nu=\beta(t)e^{\frac{1}{2}\zeta(r)}+a(r),$$

where

$$\beta=\int\alpha(t)dt.$$

Substitution into (11) shows that we can have two possibilities : (1) β is constant ; in which case we can only arrive at the statical Schwarzschild case (1) ; or (2) β is arbitrary, ζ is constant, and $a(r)$ is given by

$$\frac{d^2a}{dr^2}-\frac{1}{r}\frac{da}{dr}-\frac{1}{2}\left(\frac{da}{dr}\right)^2=0,$$

whence

$$a=-2\log\left(1+\frac{r^2}{4R^2}\right). \quad (13)$$

This merely brings us back to the cases (2) and (3). It therefore appears that there is no generalization of the Schwarzschild metric (in terms of cosmical co-ordinates) in which the mass of the Sun enters as a constant independent of the time.

Turning to solutions of (10) in which ζ is a function of both r and t, we consider first cases in which $a(r)=0$. This function is evidently dependent on the curvature of space as a whole, so that putting it equal to zero is equivalent to dealing with metrics analogous to (3). We shall show later that it is easy to extend the results we obtain to cases when $a(r)$ is of the form (13).

The solution we require must have a singularity at the origin similar to that possessed by (1). ζ and ν must therefore be expressible as power series in $1/r$. We assume

$$e^{\frac{1}{2}\zeta}=\gamma=1+\alpha_1 u^{m_1}+\alpha_2 u^{m_2}+\alpha_3 u^{m_3}+\ \ldots, \tag{14}$$

where $u=\frac{1}{r}$, the α_s are functions of t and the powers of u are arranged in ascending order. Substitution into (12) yields

$$\nu=\beta(t)+\sum_{s=1}^{\infty} u^{m_s}\beta_s(t), \tag{15}$$

where

$$\beta(t)=\int \alpha(t)dt, \tag{16}$$

$$\beta_s(t)=\int \alpha(t)\alpha_s(t)dt. \tag{17}$$

The equation (11) becomes, on changing the independent variable from r to u and substituting for ζ in terms of γ,

$$\gamma\left(u\frac{\partial^2\nu}{\partial u^2}+3\frac{\partial\nu}{\partial u}\right)+2u\frac{\partial^2\gamma}{\partial u^2}+\left(6-2u\frac{\partial\nu}{\partial u}\right)\frac{\partial\gamma}{\partial u}-\tfrac{1}{2}u\gamma\left(\frac{\partial\nu}{\partial u}\right)^2=0. \tag{18}$$

We now substitute from (14) and (15) into (17). The lowest power of u turns out to be u^{m_1-1}. On equating its coefficient to zero we obtain

$$m_1(m_1+2)(2\alpha_1+\beta_1)=0. \tag{19}$$

The next two powers of u are u^{2m_1-1} and u^{m_2-1}. Hence we have $m_2=2m_1$ and, in general, we find $m_s=sm_1$. It therefore follows from the indicial equation (19) that the only way in which we can obtain a solution as a power series in u which includes the first power of u, is by taking $m_1=1$ and

$$2\alpha_1+\beta_1=0. \tag{20}$$

This relation, together with (16) and (17), gives

$$\frac{\dot{\alpha}_1}{\alpha_1}=-\tfrac{1}{2}\dot{\beta}. \tag{21}$$

The coefficient of u^{2m_1-1} when equated to zero gives, on using (20),

$$2\alpha_2+\beta_2=\tfrac{1}{2}{\alpha_1}^2=c_2{\alpha_1}^2.$$

Differentiating this relation with respect to t and using (16), (17) again, it follows that

$$\alpha_2 = c_2\alpha_1^2 \quad \text{and} \quad \beta_2 = -c_2\alpha_1^2.$$

Proceeding in this manner it soon becomes apparent that

$$\alpha_n = c_n\alpha_1^n \quad \text{and} \quad \beta_n = -\frac{2c_n}{n}\alpha_1^n,$$

where the c_n are constants. Hence, if the solution we are seeking exists at all, it must be of the form

$$\gamma = 1 + \sum_{s=1}^{\infty} c_s\left(\frac{\mu(t)}{r}\right)^s, \tag{22}$$

$$\nu = \beta(t) - 2\sum_{s=1}^{\infty} \frac{c_s}{s}\left(\frac{\mu(t)}{r}\right)^s, \tag{23}$$

with

$$\tfrac{1}{2}\dot{\beta} = -\frac{\dot{\mu}}{\mu}.$$

In the above we have returned to the variable r and have written μ for α_1.

We can now show that γ, ν given by (22) and (23) can be expressed in finite form. For these series imply

$$\frac{\partial \nu}{\partial r} = \frac{2(\gamma-1)}{r} \quad \text{and} \quad \frac{\partial \nu}{\partial t} = -\frac{2\dot{\mu}}{\mu}\gamma\,; \tag{24}$$

hence (11) becomes

$$r^2\frac{\partial^2\gamma}{\partial r^2} - r(\gamma-1)\frac{\partial\gamma}{\partial r} - \gamma(\gamma^2-1) = 0. \tag{25}$$

We solve this equation as if γ were a function of r alone and then treat the "constants of integration" as functions of t. If we put

$$r = e^x$$

we obtain

$$\frac{d^2\gamma}{dx^2} - \gamma\frac{d\gamma}{dx} - \gamma(\gamma^2-1) = 0. \tag{26}$$

A particular first integral is obviously

$$\frac{d\gamma}{dx} = \gamma^2 - 1. \tag{27}$$

The general first integral is found as follows. Put $z = \gamma^2 - 1$ and assume $\frac{d\gamma}{dx} = z(1+w)$. Then (26) reduces to

$$z\frac{dw}{dz} = -\frac{w(2w+3)}{2(1+w)},$$

the integral of which is

$$w^2(2w+3) = \frac{A^3}{z^3}$$

or

$$\left(\frac{d\gamma}{dx}-\gamma^2+1\right)^2\left(2\frac{d\gamma}{dx}+\gamma^2-1\right)=A^3. \tag{28}$$

The expression (22) for γ involves one "arbitrary constant," $\mu(t)$, with respect to integrations by r. It must, therefore, be obtained by means of a particular first integral of (26). We notice from (28) that the particular first integral (27) is the singular solution of that equation. The particular first integral we require is, therefore, the alternative one provided by (28), viz. :

$$2\frac{d\gamma}{dx}+\gamma^2-1=0,$$

which gives

$$\gamma=\frac{1-\frac{\mu}{2}e^{-x}}{1+\frac{\mu}{2}e^{-x}},$$

$\frac{\mu}{2}$ being the "constant" of integration.

Reverting to the variable r we have

$$\gamma=\frac{1-\frac{\mu(t)}{2r}}{1+\frac{\mu(t)}{2r}},$$

where μ is now regarded as an arbitrary function of t. The first of the relations (24) then gives, on integration,

$$\nu=\beta(t)+4\log\left(1+\frac{\mu(t)}{2r}\right),$$

whilst the second is satisfied if

$$\tfrac{1}{2}\dot{\beta}=-\frac{\dot{\mu}}{\mu}. \tag{28a}$$

We can therefore say that, in terms of cosmical co-ordinates, the Schwarzschild field has the form

$$ds^2=\left(\frac{1-\mu(t)/2r}{1+\mu(t)/2r}\right)^2dt^2-\frac{1}{c^2}\left(1+\frac{\mu(t)}{2r}\right)^4e^{\beta(t)}\{dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\}, \tag{29}$$

where $\frac{1}{2}\dot{\beta}=-\frac{\dot{\mu}}{\mu}$, and μ is identified with the mass of the particle at the origin. The curvature of space is here supposed to be zero.

This result can be generalized to take account of the curvature of space when this is different from zero. For it is evident that in a small region near the origin in which the curvature of space is negligible, the field must be given approximately by (29), whilst in distant regions it must be (2). Hence γ, ν must be of the forms

$$e^{\frac{1}{2}\zeta} = \gamma = \frac{1-y}{1+y}, \qquad \nu = \beta + 4\log(1+y) + a(r), \tag{30}$$

where y is a function of (r, t) and $a(r) = -2\log(1+r^2/4R^2)$.

Substitution into (11) gives

$$\frac{\partial^2 y}{\partial r^2} - \frac{3}{y}\left(\frac{\partial y}{\partial r}\right)^2 - \frac{\partial y}{\partial r}\left(\frac{1}{r} + \frac{da}{dr}\right) = 0,$$

whence

$$\frac{1}{y^3}\left(\frac{\partial y}{\partial r}\right) = \frac{B(t)r}{(1+r^2/4R^2)^2}$$

and

$$\frac{1}{y^2} = C(t) + \frac{4R^2 B(t)}{(1+r^2/4R^2)},$$

where B, C are constants of integration with respect to r and so may be functions of t.

If we choose

$$C = 16R^2/\mu^2 \quad \text{and} \quad B = -4/\mu^2,$$

where μ is an arbitrary function of t, we obtain

$$y = \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right), \tag{31}$$

and this is easily seen to be the only value of y which reduces to $\mu/2r$ in regions where r/R is negligible. Corresponding to (31) we have, by (30),

$$\nu = \beta(t) + 4\log\left\{1 + \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}\right\} - 2\log\left(1 + \frac{r^2}{4R^2}\right), \tag{32}$$

$$\zeta = 2\log\left\{\frac{1 - \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}}{1 + \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}}\right\}. \tag{33}$$

Direct substitution of these values into (10) gives, as before,

$$\frac{\dot{\mu}}{\mu} = -\tfrac{1}{2}\dot{\beta}.$$

Thus finally : when the curvature of space is not zero, the Schwarzschild field in cosmical co-ordinates has the form

$$ds^2 = \left\{\frac{1 - \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}}{1 + \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}}\right\}^2 dt^2 - \frac{\left\{1 + \frac{\mu(t)}{2r}\left(1 + \frac{r^2}{4R^2}\right)^{\frac{1}{2}}\right\}^4}{\left(1 + \frac{r^2}{4R^2}\right)^2}\frac{e^{\beta(t)}}{c^2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \tag{34}$$

with

$$\frac{\dot{\mu}}{\mu} = -\tfrac{1}{2}\dot{\beta}.$$

It should be noted that (34) gives the field of a particle which has neither a similar particle nor a " hole " at its antipodes.* For it is obvious that as $r \to \infty$, the metric (34) tends to the form

$$ds^2 = \left(\frac{1-\frac{\mu}{4R}}{1+\frac{\mu}{4R}}\right)^2 dt^2 - \frac{\left(1+\frac{\mu}{4R}\right)^4}{\left(1+\frac{r^2}{4R^2}\right)^2}\frac{e^\beta}{c^2}\{dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\},$$

which differs from the Lemaître solution (2) only in terms of μ/R, which are negligibly small when μ is the mass of a star similar to the Sun and R is of the order of the radius of space. Hence there is no singularity as $r \to \infty$.

IV. *Planetary Systems*

We consider the reduction of the metrics found in the previous paragraphs to observer's co-ordinates. From our present point of view, we can neglect the curvature of space and take as the expression of the Schwarzschild field in cosmical co-ordinates the result (29) viz. :

$$ds^2 = \left(\frac{1-\mu(t)/2r}{1+\mu(t)/2r}\right)^2 dt^2 - \left(1+\frac{\mu(t)}{2r}\right)^4 \frac{e^{\beta(t)}}{c^2}\{dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\}. \quad (35)$$

At any instant of time t_1 the observer's co-ordinate for measuring distance from the origin is

$$r_1 = e^{\frac{\beta(t_1)}{2}} r.$$

If we write

$$m = \mu(t_1) e^{\frac{\beta(t_1)}{2}},$$

the above metric becomes

$$ds^2 = \left(\frac{1-m/2r_1}{1+m/2r_1}\right)^2 dt^2 - \frac{1}{c^2}\left(1+\frac{m}{2r_1}\right)^4 \{dr_1^2+r_1^2(d\theta^2+\sin^2\theta d\phi^2)\}, \quad (36)$$

and thus appears to the observer, at the instant t_1, as the usual statical field (1). But, in virtue of (28*a*), the derivative of m as a function of t is zero. Hence the reduction of (35) to (36) is independent of the instant in time at which the transformation is made and the field of the mass-particle will therefore be considered as strictly statical by the observer.

Again, if we compute the components, $(T_{ik})^*$, of the energy tensor for the matter outside the mass-particle in terms of the observer's system of co-ordinates (r_1, t, θ, ϕ) we find

$$\left.\begin{aligned}(T_4^4)^* = T_4^4 = \rho;\ (T_1^1)^* = T_1^1 = -p_1;\ (T_2^2)^* = (T_3^3)^* = T_2^2 = T_3^3 = -p_2\\ (T_{14})^* = e^{-\frac{\beta(t_1)}{2}} T_{14}.\end{aligned}\right\} \quad (37)$$

* This is in contrast to all the cases previously studied by Dr. W. H. McCrea and the author. See *M.N.*, **92**, 7–12, 1931.

Hence, since $p_1 = p_2$ and $T_{14} = 0$, the conditions (a) to (f) of paragraph II all hold in the observer's system. He concludes that the system of nebulæ is "at rest" in his system in the sense explained in the Introduction. He is therefore justified in using a set of cosmical co-ordinates, as we have done, when it is mathematically convenient. The use of such a system, it should be noticed, does not imply the existence of a "cosmical time" common to all observers in the universe.* For we can suppose the observer to set up first of all the system (36) in which he chooses the "time" to suit his local circumstances without reference to the "times" of other observers. It is then this purely local time that he employs if he decides to introduce the cosmical co-ordinates in which the metric takes the form (35).

In the case of our Sun, the terms m/r_1 in (36) are already negligibly small at the distance of the Earth. If our observer is near a mass-particle of stellar mass, the metric (35) rapidly becomes indistinguishable from (2) as we recede from the origin. The observer will thus be able to account for the "recession" of a distant nebulæ by assigning a fixed value to the cosmical co-ordinate r of the nebula and interpreting the red-shift of the lines of the spectrum as due to the change in the co-ordinate r_1 during the time the light travels to him from the nebula. In this way he finds that, at the present moment,

$$\dot{\beta} = 2 \times 10^{-17} \text{ sec.}^{-1}.$$

We shall now consider planetary orbits in terms of the cosmical co-ordinates in which the metric is (35). There are, of course, the closed geodesics around the origin. The equations of any geodesic are

$$\frac{d^2x^\alpha}{ds^2} + \{\mu\nu, \alpha\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (\alpha = 1, 2, 3, 4),$$

where x^α stands for any one of (r, t, θ, ϕ) and the $\{\mu, \nu, \alpha\}$ are the Christoffel symbols with respect to (35). Direct calculation of the equation for θ shows that if, initially,

$$\frac{d\theta}{ds} = 0, \qquad \frac{d^2\theta}{ds^2} = 0, \qquad \theta = \tfrac{1}{2}\pi,$$

then the geodesic will always be in the plane $\theta = \frac{1}{2}\pi$. The problem of determining the orbit of the planet thus reduces to finding the (r, ϕ) equation of the geodesics in a plane whose metric is

$$ds^2 = e^{\zeta(r,t)}dt^2 - \frac{e^{\nu(r,t)}}{c^2}\{dr^2 + r^2d\phi^2\}, \tag{38}$$

where ν, ζ are given by

$$\zeta = 2\log\left(\frac{1-\mu/2r}{1+\mu/2r}\right), \qquad \nu = \beta(t) + 4\log(1+\mu/2r). \tag{39}$$

To do this we write

$$\sigma = is, \qquad \tau = it,$$

* This has been contended by E. A. Milne, *Nature*, **130**, 9–10, 1932 July 2, and pp. 507–508, 1932 October 1.

so that (38) can be written

$$1 = e^{\zeta(r,\tau)}\left(\frac{d\tau}{d\sigma}\right)^2 + \frac{e^{\nu(r,\tau)}}{c^2}\left\{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2\right\}. \tag{40}$$

We put

$$L = \tfrac{1}{2}\left[e^{\zeta}\left(\frac{d\tau}{d\sigma}\right)^2 + \frac{e^{\nu}}{c^2}\left\{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2\right\}\right].$$

We can then regard L as the kinetic potential of a dynamical system moving under no forces, in which σ plays the part of the time and (40) is the integral of energy. Our problem is then equivalent to finding the equations of motion of this system with ϕ instead of σ as independent variable, using the integral of energy (40). The details of the method will be found elsewhere.* The result is that the equations of motion are expressible in terms of a function

$$L' = \left[e^{\zeta}\left(\frac{d\tau}{d\phi}\right)^2 + \frac{e^{\nu}}{c^2}\left\{\left(\frac{dr}{d\phi}\right)^2 + r^2\right\}\right]^{\frac{1}{2}} \tag{41}$$

in the usual form

$$\frac{d}{d\phi}\left(\frac{\partial L'}{\partial r'}\right) - \frac{\partial L'}{\partial_r} = 0\,; \qquad \frac{d}{d\phi}\left(\frac{\partial L'}{\partial \tau'}\right) - \frac{\partial L'}{\partial \tau} = 0, \tag{42}$$

where

$$r' = \frac{dr}{d\phi}, \qquad \tau' = \frac{d\tau}{d\phi}.$$

An integral of these equations is

$$\tau'\frac{\partial L'}{\partial \tau'} + r'\frac{\partial L'}{\partial r'} - L' = -\frac{ih}{c}, \tag{43}$$

where h is a (real) constant. This equation becomes, on substituting from (41) and squaring both sides,

$$e^{\zeta}\left(\frac{d\tau}{d\phi}\right)^2 + \frac{e^{\nu}}{c^2}\left\{\left(\frac{dr}{d\phi}\right)^2 + r^2\right\} = -\frac{r^4 e^{2\nu}}{h^2 c^2}. \tag{44}$$

On using (40) and returning to the variables t, s we obtain

$$r^2 e^{\nu(r,t)}\frac{d\phi}{ds} = hc, \tag{45}$$

so that (43) is the "integral of angular momentum" for the motion of the planet in the field (35).

The equation of the orbit is the first of the equations (42). If we put $u = 1/r$ and use (44) we get, after some reduction,

$$\frac{d^2u}{d\phi^2} + u = -\frac{e^{\nu(r,t)}\zeta_u}{2h^2} + \frac{u^2(\nu_u - \zeta_u)}{2} - \frac{(\zeta_u - \nu_u)}{2}\left(\frac{du}{d\phi}\right)^2, \tag{46}$$

where ν_u, ζ_u denote derivatives with respect to u.

* E. T. Whittaker, *Analytical Dynamics*, § 42, 3rd ed., 1927.

We consider the "Newtonian" approximation to (46) when ζ, ν are given by (39). The result is

$$\frac{d^2u}{d\phi^2}+u=\frac{e^{\beta(t)}\mu(t)}{h^2}. \tag{47}$$

If now the planet concerned has a period short compared with the rate of change of β and moves with a velocity small compared with that of light, we can take the integral of (47) to be

$$u=\frac{e^{\beta(t)}\mu(t)}{h^2}(1+\epsilon\cos\phi), \tag{48}$$

where ϵ is the eccentricity of the orbit. Suppose ϵ is fixed; then the value of r, for a given ϕ, will change at a rate

$$\frac{1}{r^2}\frac{dr}{dt}=-\frac{e^{\beta(t)}\dot{\beta}\mu}{2h^2}(1+\epsilon\cos\phi)$$

since $\dot{\beta}=-2\dot{\mu}/\mu$, and h is an absolute constant by (45). Also $\dot{\beta}$ is positive. Thus in terms of cosmical co-ordinates the orbit of the planet continually shrinks and, from (28*a*), it follows that the mass of the central particle continually decreases. Hence, from the point of view of cosmical co-ordinates, the configuration of the nebulæ is unchanging, but the mass-particle and its planetary system perpetually shrink in size.*

On the other hand, from the observer's point of view, the orbit of the planet is (at the instant t_1)

$$\frac{1}{r_1}=\frac{m}{h^2}(1+\epsilon\cos\phi)$$

and the mass, m, of the central particle is constant. Hence, for the observer, the orbit of the planet and the mass of the central particle remain fixed, whilst the system of the nebulæ increases in size.

It should be remarked that this result turns on the relation (28*a*) which itself follows from the assumption that the system of the nebulæ has, on the whole, zero momentum in the observer's system. If, as has been suggested by E. A. Milne,† the "recession" were due to the nebulæ having true co-ordinate velocities directed away from the observer, the component (T_{14})* of the energy-tensor could certainly not be zero in the observer's co-ordinate system. It is difficult to see how the "recession" could then fail to have observable repercussions on planetary systems,‡ even if the serious difficulty of endowing the nebulæ with sufficient momentum to produce velocities corresponding to the observed enormous red-shifts is overlooked.

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* A. S. Eddington, *Phys. Soc. Proc.*, 44, 1–17, 1932 January 1.

† E. A. Milne, *loc cit.*

‡ See in this connection G. C. McVittie, *M.N.*, 92, 500–518, 1932, where the theory of a particle with constant mass in the cosmical co-ordinate system is worked out.

V. *The Pressure and Density and the Cosmical Constant*

Returning, for a moment, to cosmical co-ordinates we calculate the density and pressure outside the mass-particle in the field (34). Since the pressure is now isotropic, we put $p_1 = p_2 = p$. Solving the equations (6) and (7) or (8) for ρ and p, we obtain

$$\kappa c\rho = -\lambda + \tfrac{3}{4}e^{-\zeta}(\dot{\nu})^2 - c^2 e^{-\nu}\left\{\nu'' + \frac{2}{r}\nu' + \tfrac{1}{4}(\nu')^2\right\}, \tag{49}$$

$$\kappa cp = \lambda - e^{-\zeta}\{\ddot{\nu} + \tfrac{3}{4}(\dot{\nu})^2 - \tfrac{1}{2}\dot{\zeta}\dot{\nu}\} + \frac{c^2 e^{-\nu}}{4}\left\{\zeta'' + \nu'' + \frac{3}{r}(\zeta' + \nu') + \tfrac{1}{2}(\zeta' + \nu')^2\right\}, \tag{50}$$

where ζ, ν are given by (33) and (32) respectively.

Suppose we denote by ρ^* and p^* the values of the density and pressure calculated in the observer's system at the instant t_1. The result (37) shows that ρ^*, p^* are obtained from ρ, p simply by the substitutions $r = r_1 e^{-\frac{\beta(t_1)}{2}}$, $\mu = me^{-\frac{\beta(t_1)}{2}}$. Using (32), (33), (49) and (50) we obtain, after some calculation,

$$\kappa c\rho^* = -\lambda + \tfrac{3}{4}\dot{\beta}_1^2 + \frac{3c^2}{R^2}\left\{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\right\}^{-4} - \frac{3c^2}{R^2}\frac{m}{r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\left\{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\right\}^{-5},$$

$$\kappa cp^* = \lambda - \frac{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}}{1 - \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}}\ddot{\beta}_1 - \tfrac{3}{4}\dot{\beta}_1^2 - \frac{c^2}{R^2}\left\{1 + \frac{m}{2r_1}\left(1 + \frac{r_1^2}{4R_1^2}\right)^{\frac{1}{2}}\right\}^{-4},$$

where $\ddot{\beta}_1$, $\dot{\beta}_1$ are the values $\ddot{\beta}$, $\dot{\beta}$ at the instant t_1 and $R_1 = e^{\frac{\beta(t_1)}{2}} R$ is the radius of space at that moment.

It has been suggested by certain investigators * that the cosmical constant, λ, which appears in these formulæ is merely a mathematical device and that its value is indifferent from the physical point of view. They propose putting $\lambda = 0$. This cannot be done, however, without introducing difficulties with regard to the expansion. For suppose $\lambda = 0$. Then it follows, since ρ^*, p^* cannot be negative, that $\ddot{\beta}_1 < 0$ whatever $\dot{\beta}_1$ and $1/R^2$ may be. Moreover, this holds for all instants t_1. But our observer has found from his observations of the nebulæ that $\dot{\beta}_1 > 0$. Thus he must conclude that the expansion is proceeding subject to a retardation. Exactly the same result follows if $\lambda < 0$. Hence, in either case, he must conclude, firstly, that at some time in the past the expansion started instantaneously with a finite velocity; secondly, that there is a "retarding force" slowing up the expansion which, obviously, cannot be the initial cause that started the

* A. Einstein, *Preuss. Akad. Wiss. Berlin, Ber.*, **12**, 235–237, 1931; A. Einstein and W. de Sitter, *Nat. Acad. Sci. Proc.*, **18**, 213–214, 1932.

latter. No attempt has been made to account for these properties of the expansion, nor is it easy to imagine how they could arise. We are driven to the conclusion that our observer would necessarily take λ to be a positive constant. If he does so he is at once presented with a familiar type of mechanism for starting the expansion: the disturbance of the equilibrium (Einstein) state of the universe. As Eddington has shown,* $\ddot{\beta}$ and $\dot{\beta}$ are then always of the same sign and the effect of the disturbance is cumulative, $\ddot{\beta}$ increasing with $\dot{\beta}$ if $\ddot{\beta} > 0$ initially.

VI. *Equilibrium of Einstein Universe*

It is necessary to point out that the new metric (34) for a mass-particle in a spherical universe of initial curvature $1/R^2$ does not throw any new light on the question of the disturbance of the equilibrium of the Einstein universe by the formation of condensations. As in previous cases considered,† the volume of the new universe, for constant proper mass, is identical with that of the corresponding Einstein universe, to a first approximation. No method for dealing with the second approximation has yet been discovered. An alternative method is proposed by Narliker,‡ who suggests that the formation of a condensation is accompanied by an increase in the cosmical constant, λ, from which he deduces that the Einstein universe would begin to expand. A serious objection to this is that the universe after the condensation has formed no longer corresponds to the same Einstein universe as that in which the process of condensation began.§ Because of the increase of λ, the proper-mass of the whole universe has decreased. Narliker's treatment fails to show that it is the formation of the condensation and not this change of proper-mass which actually starts the expansion. It must be remembered, of course, that a decrease of proper-mass, without a corresponding adjustment of the radius, of an Einstein universe will of itself start an expansion.‖ The problem, in our opinion, is to determine whether or not an Einstein universe of given proper-mass will expand or contract when a condensation forms, the proper-mass being conserved throughout all changes.

* A. S. Eddington, *M.N.*, **90**, 668–678, 1930.

† W. H. McCrea and G. C. McVittie, *M.N.*, **92**, 7–12, 1931.

‡ V. Narliker, *Phil. Mag.* (7), **14**, 433–436, 1932.

§ The relation between the proper-mass, M, of an Einstein universe and λ is $M = \frac{\pi}{2\sqrt{\lambda}}$. Hence to each value of λ there corresponds an Einstein universe of different proper-mass, radius, etc.

‖ A. S. Eddington, *M.N.*, **90**, 668–678, 1930.

TOTAL ECLIPSE OF THE SUN, 1932 AUGUST 31: REPORT OF THE IMPERIAL COLLEGE EXPEDITION.

Herbert Dingle, D.Sc.

Introductory

The purpose of this expedition was twofold. First, to photograph the bright-line spectrum at the cusp of the partially eclipsed Sun, as observed visually by Professor A. Fowler * and photographed by Professor Newall † at the eclipse of 1912 April 17. Secondly, to photograph and measure the Fraunhofer spectrum near the limb of the Sun just before and after totality, when the diffused atmospheric light, being also limb light, could have no disturbing influence on the wave-lengths obtained.

Both problems required large dispersion in the spectrograph employed, and it was proposed to use the same equipment, photographing the cusp spectrum up to a few minutes before and subsequent to a few minutes after totality, and the limb spectrum in the periods immediately before and after totality. Since for these investigations a long period of totality—or, indeed, totality at all—was unnecessary, the choice of stations was much wider than is usual in eclipse work, and consideration of convenience in such matters as the availability of dark-room accommodation, supply of electricity and ordinary laboratory facilities, could be given much weight. The circumstance that the edge of the shadow belt passed near Montreal naturally suggested McGill University, and kind offers of assistance from Dr. A. S. Eve, Dean of the Graduate Faculty, led to this institution being chosen for the work. In view of the fact that helpers were available at the University, it was considered unnecessary for more than one observer to go out from England. Leaving Southampton on S.S. *Montcalm* on July 20, Montreal was reached on July 28, thus allowing nearly five weeks for the preliminary work.

The expedition was one of those arranged by the Joint Permanent Eclipse Committee, for which funds were provided by the Government Grant Committee of the Royal Society. Valuable assistance was also given by the Imperial College in the form of workshop facilities and in numerous other ways.

It is unfortunate that no results were possible on account of unfavourable weather, but valuable experience was obtained.

Apparatus

The apparatus prepared was similar in principle to that arranged by Professor Fowler for the eclipse of 1914 and described in *M.N.*, 75, 117, 1915. The details, however, were almost completely different. For

* *M.N.*, 72, 538, 1912. † *Ibid.*, 536, 1912.

purposes of description the apparatus may conveniently be divided into two parts: (*a*) the spectrograph; (*b*) the arrangements for producing images of the Sun and of the iron arc used to give a comparison spectrum.

(a) *The Spectrograph.*—The arrangement finally adopted consisted of two 60° glass prisms mounted successively at minimum deviation, backed by a plane Rowland grating. The grating was mounted so as to return through the prisms, as in the Littrow arrangement, the light diffracted in a first-order spectrum, the effective dispersion thus being approximately that of four 60° prisms plus the first order of the grating. The lens used was a

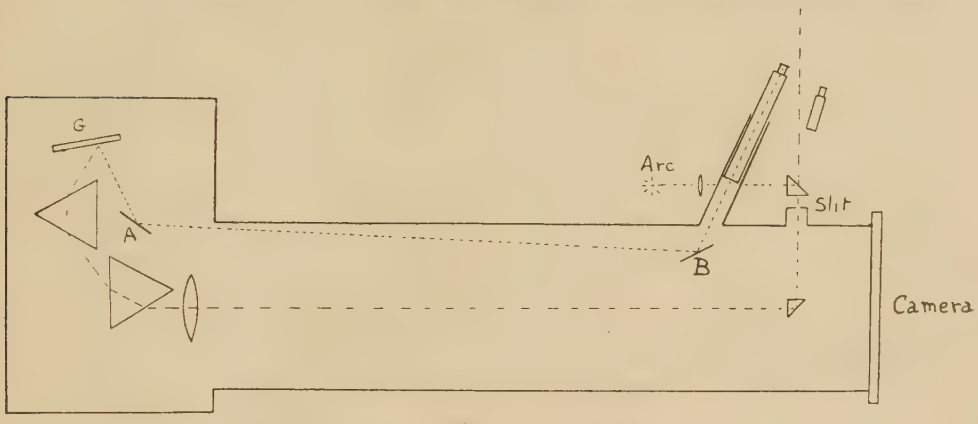


Fig. 1.

4-inch Cooke photo-visual object-glass with a focal length of 16 feet, and the resulting spectrum in the selected region (λ 4200 to λ 4400) had an average dispersion of about 0·8 A. per mm.

In the cusp problem it was desirable to be able to observe the spectrum visually in order to ensure that bright lines were actually being photographed. To provide for this, light directly reflected from the grating was used in the way indicated in fig. 1. A plane mirror placed at A sent such light (which had already received dispersion in the two prisms) down the spectrograph case to a second plane mirror B, whence it was sent through a tube in the side of the case to an observing telescope whose eyepiece was adjacent to that of a small telescope used for observing the image on the slit. The eye could thus be readily moved from one telescope to the other. Both mirrors A and B were capable of rotation about horizontal and vertical axes, so that light in any part of the spectrum could be directed accurately along the axis of the viewing telescope. The region including $H\alpha$ was selected for observation, and to prevent the admission of troublesome stray light the eyepiece was covered by a small piece of ruby glass.

(b) *Solar and Arc Images.*—For the purpose in view it was essential to provide some means whereby the image of the Sun could be rotated, so that the cusp could be kept on the slit of the spectrograph during the progress of the eclipse. A 12-inch cœlostat directed the rays of the Sun in an approximately horizontal direction to a second 12-inch mirror, which could be adjusted in two directions, the adjustment about a horizontal axis being

controlled by the observer of the spectrograph slit with the aid of a long rod. The second mirror reflected the rays to a 6-inch achromatic object-glass of 78 inches focal length. Lenses for enlarging the image and a reflecting prism for rotating it were combined in a single cylindrical holder. On the objective end of this was an achromatic diverging lens of $9\frac{1}{2}$ cm. focal length, placed in such a position as to make the rays parallel, in which condition they passed through the prism, as shown diagrammatically in fig. 2. The prism (90° angle) was truncated, and designed to invert the image without lateral displacement. After passing through the rotator the rays were brought to

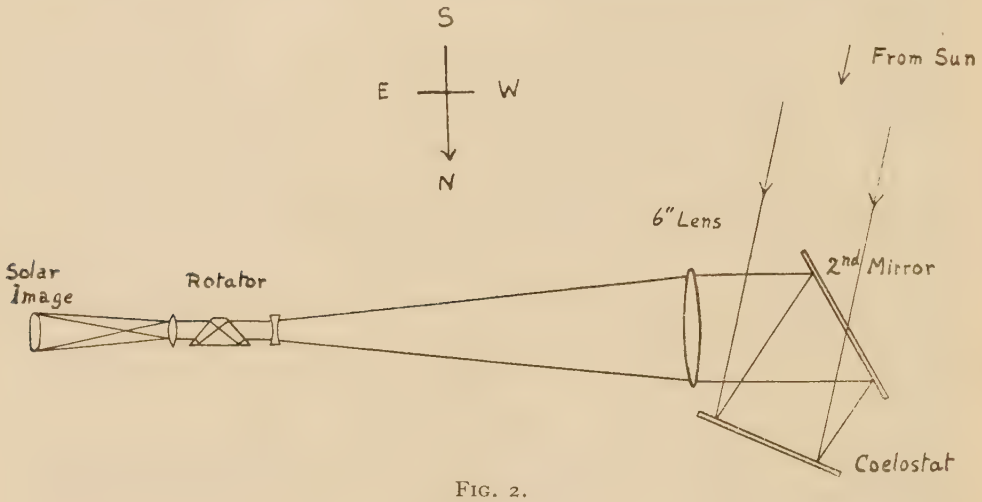


Fig. 2.

a focus on the slit by a converging lens of 50 cm. focal length. The 6-inch objective alone would give a solar image of 0·68 inch, and the negative lens, being $50/9\frac{1}{2}$ times as powerful as the succeeding positive lens, enlarged this to a diameter of $3\frac{1}{2}$ inches.

With regard to the rotator, it should be noted that the rays passing through it should be parallel in order that it may produce no dispersion. Also, if the rotator is not to deflect the rays laterally, the length of path of the rays internally must bear a definite relation to its mean refractive index. When the length is correct, an incident pencil symmetrical about the axis is transmitted without lateral displacement as a whole, but rays of the pencil equidistant from the axis on opposite sides of it are interchanged in position. The image, as is well known, rotates at twice the rate of the prism. All necessary adjustments of the rotator were suitably provided for. The turning of the rotator was done by hand, and its position at any time was indicated on a scale of degrees. On a future occasion, however, it would be advisable to provide a slow-motion screw for the rotator.

It is clear that the best conditions in the cusp problem would be obtained by keeping the slit tangential to the image of the Moon at the cusp rather than to that of the Sun, since a greater length of slit would then be illuminated by the chromosphere and reversing layer. To secure this the cœlostat should be

rated so as to keep the Moon's image stationary, and the rotator adjusted to turn the image about its centre.

On the other hand, to obtain the Fraunhofer limb spectrum it was necessary to keep the image of the Sun stationary. The alteration from the rate of the driving clock for the Sun to that for the Moon during the eclipse having been calculated, the positions of the rating screw for both Sun and Moon were carefully marked, so that it would be easy to change from one to the other at a moment's notice.

The actual adjustment of the various optical parts for forming the image on the slit was somewhat complicated, but it is not considered necessary to give the full details of the operations here.

The iron arc for the comparison spectrum was produced at the side of the spectrograph, and its image was projected on the slit by means of a converging lens and reflecting prism, as shown in fig. 1 ; the reflecting prism was so mounted that it could readily be brought into its correct position in front of the slit when required, and as readily removed. In order to produce more lines than the iron arc alone would give, trials were made of an iron-tungsten arc, but as this source was too unsteady to ensure the absence of pole effect, it was decided to use iron alone.

The mountings of all the optical parts of the equipment except the cœlostat were constructed in the Imperial College workshops from designs by Professor A. F. C. Pollard, Professor of Instrument Design, and were satisfactory in every way.

Conditions of the Eclipse

Through the kindness of Dr. L. J. Comrie, Superintendent of the *Nautical Almanac*, data were provided giving the position-angles of the cusps, as seen from McGill University Observatory, at intervals of one minute for a period of 35 minutes on either side of totality. These position-angles were useful for determining the change in Doppler rotation effect during an exposure, and it appeared that, excluding 5 minutes on either side of totality, exposures up to 10 minutes in duration could have been given without broadening the spectrum lines by more than 0·003 A.

From the data given by Dr. Comrie, position-angles of the cusps as measured along the Moon's limb were calculated, and it was found that the movement of the cusps, except near totality, would be very slow. Since the rotator has to be moved only half as fast as the image, no difficulty was anticipated in following the cusp during an exposure.

The length of the slit to be illuminated by the light of the part of the reversing layer protruding at the cusp could be calculated from the following formula given by Dr. Comrie :—*

$$\left.\begin{array}{l}\text{Angle subtended at}\\ \text{Moon's centre by}\\ \text{overhang of revers-}\\ \text{ing layer}\end{array}\right\} = \frac{0{\cdot}5459\,S}{2\sin 1^{\circ}\, m \sin A \times \text{Moon's semi-diameter}} \text{ degrees,}$$

* *M.N.*, 87, 485, 1927. In this paper the Sun's semi-diameter appears in the denominator instead of the Moon's, through an error.

where S = depth of reversing layer in seconds of arc, m and A have their usual meanings in such computations, and the Moon's semi-diameter is to be reckoned in seconds of arc.

Thus, with a reversing layer 2 seconds in depth and a solar image of $3\frac{1}{2}$ inches in diameter, the length of slit illuminated would have been 0·12 mm. at 30 minutes before totality, 0·30 mm. at 10 minutes, 0·55 mm. at 5 minutes and 1·2 mm. at 2 minutes before totality. These are smaller than might be considered desirable, but the experience gained by Professor Fowler in 1912 suggests that, with careful guiding, they would probably have been adequate for the purpose in view. The large solar image thus rendered necessary, combined with absorption in the rotating system, would have given a fainter spectrum than was desired, but an effort would have been made to make the best use of the available light by employing very sensitive photographic plates. Mr. Olaf Bloch of the Ilford Company kindly supplied plates coated with a non-red-sensitive emulsion at least twice as rapid as that of the Ilford "Ordinary" plates; they were of plate-glass, 12 inches × 2 inches, and backed.

Choice of Site

After careful examination of several possible sites, the roof of the Molson Hall of the McGill University was selected. This gave an excellent view of the Sun, and subsequent experience showed that a very steady solar image was possible, despite large variations of temperature. The supply of 110 V. D.C. could be brought to the roof by cable from the rooms below, and an excellent dark room, also just below the roof, was available. To save time an arrangement was made for lowering the plate-holders to the window of a room adjoining the dark room.

On account of the construction of the roof, it was found necessary to distribute the load of the instruments, and this was effected by making a base of long horizontal beams of wood, with cross-pieces to prevent torque.

The spectrograph was set up with the collimator in a north and south direction. Piers of concrete and steel piping were erected for the dispersing and camera ends of the instrument, and a wooden case was made to enclose the whole of it, suitable openings being provided for purposes of adjustment. The effects of large variations of temperature were eliminated by providing a double-walled cover at the dispersing end of the apparatus, in the interior of which air could circulate, and covering the outside with "tentest," a material of very low conductivity. The whole instrument was further covered by a tarpaulin to protect it from the direct rays of the Sun and from rain.

Proposed Procedure During Eclipse

The nature of the programme demanded a procedure essentially different from that usually adopted in eclipse work, as the observations were to have lasted for the greater part of an hour. Since the length of exposure required for the cusp spectrum was quite unknown, it was intended, as soon as bright

lines were visible through the small observing telescope, to give a short exposure, send the plate down for immediate development, and to arrange the subsequent work accordingly. This necessarily left an uncertainty in the programme which could not be removed.

In view of these conditions no attempt was made to assign to each helper a definite piece of work at a definite moment. The whole programme had, in fact, to be adjusted by the writer according to the exigencies of the moment. Rehearsals, allowing arbitrary times for the exposures, however, sufficed to show that this procedure was practicable.

The programme contemplated, on the assumption of a cloudless sky, was briefly as follows. As soon as bright lines were visible an exposure of 3 minutes was to be given, and the plate developed immediately and examined; then, before totality, one exposure to the cusp spectrum and one to the limb spectrum. After totality these two exposures were to be repeated in inverse order, and, if time permitted, an additional exposure was to be given to the cusp. Each spectrum was to have been provided with at least one, and, if time permitted, two comparison spectra of iron. The programme was arranged so as to allow every worker to watch the Sun during totality, when no scientific observations were to be undertaken.

On the morning of the eclipse day the weather was not at all promising, and as the day progressed the sky became more and more cloudy; consequently no observations during the eclipse were possible.

Acknowledgments

Thanks are due to numerous helpers whose assistance was freely offered and proved of the greatest value. In the preliminary work at South Kensington, besides the assistance already mentioned, the advice and experience of Professor Fowler were essential elements in the planning of the expedition.

At Montreal Dr. A. S. Eve placed the resources of the Physics Department, and Dean Ira Mackay those of the Arts Building, at the disposal of the expedition, and the former gave invaluable help in numerous ways during the preparations. Among others who gave ready and ungrudging assistance, and were prepared to take part in the work during the eclipse, were Professor F. M. Wood, Professor A. J. Kelly, Dr. J. J. Green, Dr. J. F. Heard, Mr. C. H. Skelton, Mr. S. M. Neamtan, Mr. S. Amesse, Mr. Tweedale and Mrs. Dingle. Professor Wood and Mr. Neamtan shared in the daily labour of preparation during the weeks preceding the eclipse and in the subsequent packing of the instruments. Mr. Macfarlane, Superintendent of Works at McGill University, provided assistance from the workshop throughout the erection of the apparatus. Dr. Otto Struve, Director of the Yerkes Observatory, kindly telegraphed the position-angles of prominences on the Sun's limb on the morning of the eclipse day. Mr. C. R. Davidson, of the Royal Observatory, Greenwich, superintended the conveyance of the instruments from the docks at Montreal to South Kensington, as the writer had arranged to remain in America for some months after the eclipse.

DETERMINATIONS OF EFFECTIVE WAVE-LENGTHS AND COLOUR INDICES OF EROS AND THE SOUTHERN PRIMARY REFERENCE STARS, AND THEIR APPLICATION TO THE CONTROL OF SYSTEMATIC ERRORS IN DETERMINATIONS OF THE SOLAR PARALLAX DUE TO ATMOSPHERIC DISPERSION.

H. Horrocks, M.A.

(Communicated by H.M. Astronomer)

1. *Introduction.*—In connection with the observation of Eros at the opposition of 1900–1901, for the determination of the solar parallax, Henry compared the effective wave-length of Eros with the mean of the effective wave-lengths of the comparison stars in two fields by means of photographs taken with a coarse wire grating in front of the objective. From the one plate Eros appeared bluer, and from the other redder, than the mean of the comparison stars, the differences amounting to 79 A. and 36 A. respectively.* Adopting a formula for the refractive index of air given by Kayser and Runge in their *Handbuch der Spectroscopie*, Dyson has shown that a systematic difference of only 10 A. between the effective wave-lengths of Eros and the comparison stars would introduce an error of 0″·0043 into the value of the solar parallax derived from the Greenwich series of observations.† From these results it is clear that a careful investigation of the effects of atmospheric dispersion on the observed positions of stars is essential if such systematic errors are to be controlled in the reduction of the observations made in 1930–1931. The object of such an investigation is the determination of the laws governing the differential displacements of comparison stars and Eros as observed photographically or visually at varying zenith distances, expressed in terms of any convenient colour-parameter, in order that the final selection of reference stars may be so made in relation to this parameter that the influence of systematic errors of this nature may be minimised.

Photographs using a coarse wire grating in front of the objective giving first-order diffraction images with dispersion of the same order of magnitude as atmospheric dispersion at 60° Z.D. (but rather exceeding the latter) are most suited to this undertaking. The factors which determine the apparent displacements due to refraction are the same as those which determine the measured distances Δ between the first-order images when the latter are not too unlike stellar images in appearance, and the rather higher dispersion of the grating will reduce the influence of errors due to measurement. Within the short range of wave-lengths with which we are effectively concerned

* *Com. Int. Perm. de la Carte du Ciel*, Circulaire No. 8, 1901.

† *M.N.*, 72, 420, 1912.

we may assume the atmospheric dispersion to be linear as well as the dispersion of the grating. Under these circumstances the variation from star to star of the refraction effect at a given zenith distance is proportional to the variation of Δ from star to star, provided that the observations of the latter were made with the same instrument under comparable conditions. If, then, the mean of Δ for all comparison stars used agrees with the value of Δ for Eros, systematic error due to colour will be eliminated from the final result for the solar parallax.

2. *Limited Applicability of Results; Nature of Images; Reduction to Uniform System.*—As has been pointed out by others, determinations of effective wave-lengths by the grating method are markedly influenced by the optical properties and adjustments of the telescope employed and also by the photographic plates used,* as well as by the effects of atmospheric absorption. The results may therefore be applied only with caution to observations with other instruments and plates.

The nature and extension of a diffraction image near the optical axis of the instrument, and hence its measured position, depend in the first place on the photographic efficiency of the light of different wave-lengths falling on the plate, and the minimum value required to give a perceptible density of deposit. On account of the troublesome magnitude equation arising in this way from the unequal growth towards the red and the blue of an image due to a source whose intensity is increased without change of quality, it has been usual to give effective wave-lengths referred to a standard image intensity. This type of magnitude equation, however, must also be inherent in displacements due to atmospheric dispersion. For our present purpose it is therefore preferable to adopt as our system of effective wave-lengths that given by a standard exposure time so chosen as to give first-order images comparable with images on the plates of the solar parallax series. Corrections to reduce the results of other exposure times to the standard may be determined observationally. In the system adopted here, a determination of Δ under standard atmospheric and instrumental conditions will depend upon

(*a*) The distribution of the energy in the stellar spectrum, *i.e.* the stellar temperature, assuming Planck's law.

(*b*) The apparent magnitude.

On account of differential atmospheric absorption, Δ needs correction to the zenith if it is to serve as a colour parameter based on uniform conditions of exposure.

It is shown by Lindblad that for images formed away from the optical axis oblique incidence on the grating does not appreciably affect the angular separation of the first-order spectra.† Such factors, however, as imperfect achromatism, curvature of the field and coma will affect the form of the images considerably. Corrections of considerable magnitude have been

* See, *e.g.*, Lundmark and Luyten, *M.N.*, **82**, 495, 1922; Dyson and Martin, *M.N.*, **86**, 28, 1925.

† *Ark. Mat. Astr. Fys.*, **13**, No. 26, 1918.

found necessary to reduce to the optical axis measures of Δ at points away from the centre of the plate.

3. *Instrument, Grating and Observational Material.*—The grating used in conjunction with the astrographic telescope was constructed in the observatory workshop by stretching wire of 0·065 in. diameter across a metal frame designed to fit on to the object-glass cell and oriented to give dispersion in the R.A. direction. The widths of bar and space are closely equal, an overall measurement of the grating giving 0·1235 in. ≡ 0·3137 cm. as the average width of bar + space. The dispersion of such a grating is 0·006575 seconds of arc per Angström unit as compared with 0″·00083 tan z for atmospheric dispersion.* Two exposures each of 10 minutes were given under normal atmospheric conditions, and the attempt was made to avoid nights when the conditions appeared inferior for work on this programme. In order to complete the programme within the same period as other Eros observing, however, nights of inferior quality had to be employed and the normal exposure time was sometimes increased to 12 minutes. Large variations from night to night are known to arise in effective wave-length determinations, which are not readily correlated with the apparent transparency of the sky. Such variations are confirmed by the present series of observations, and will give rise to accidental errors of much greater importance than those due to the above changes of exposure time. Oberguggenberger has recently found that these variations have a marked correlation with temperature changes.† They are probably due to the effect of temperature on the focus.

The path of Eros south of +16° dec. was covered with overlapping plates so distributed that each reference star occurs on two, three or four plates. Observing was restricted to hour angles not greater than $1^h\ 15^m$, and the mean hour angle of a plate very rarely exceeds 1 hour. The results of the present paper are based on the measurement of a selection of 45 plates south of the equator, on which almost all the stars for which Seares, Sitterly and Joyner have given magnitudes and colour indices are found at least once.‡

"Monarch" plates have been employed, and the measurement was carried out on a Repsold machine using the same portion of the screw throughout. The unit of measurement of the distance Δ between the first-order images is 0·5 mm. As the scale of the instrument is 1 mm. = $1 - \frac{1}{300}$ minutes of arc, the conversion of Δ into effective wave-length is given by

$$\lambda = 2274\Delta.$$

4. *Measurement of Plates.*—The programme of measurement included all the stars of Kopff's three lists of primary reference stars, and all measurable stars contained in the Henry Draper Catalogue. All plates have been measured in the direct and reversed positions, some by the writer and others by Mr. Menzies, the agreement between us being good. When gradation of density could be detected, the settings on the diffraction images were made on the point of maximum density. Except for the brighter stars, an un-

* Dyson, *loc. cit.* † *Mitteilungen der Sternwarte Innsbruck*, Nr. 5, 1931.
‡ *Ap. J.*, 72, 311. 1930.

symmetrical gradation was not discernible and the settings became bisections. The settings on the images of very bright stars are subject to great uncertainty, and the brightest stars have been excluded. The diameters of the central images were also measured as criteria of apparent magnitude.

The programme of measurement also included a bisection of the central image in order to examine separately the diffraction displacements Δ_1, Δ_2 of the following and preceding first-order images. Each was found to contribute equally to the correction to the centre of the plate, and to the probable error of Δ as derived from the comparison of observations on different nights.

The probable error of a single determination of Δ (the mean of two exposures each measured in both positions of the plate) due to error of setting only is found from a comparison of direct and reversed measures to be 0·0050. The value 0·0035 was obtained from a comparison of the individual exposures on a selection of 12 plates taken at random. This suggests that plates obtained under the same conditions of exposure and development are not affected by errors, such as guiding error, to an extent comparable with the error of setting.

5. *Correction to Plate Centre.*—When the same star occurs on n different plates in positions (x', y'), (x'', y''), etc., the $n-1$ independent differences that can be formed from the measured Δ's give rise to equations of the form

$$c(x', y') - c(x'', y'') = \Delta(x'', y'') - \Delta(x', y'),$$

in which $c(x, y)$ is the correction to be applied to a measure of Δ at (x, y) in order to reduce to the centre of the plate.

The area of measurement was divided into thirty-six equal squares, and as a first approximation the correction was assumed to be constant throughout each square. After the solution of the normal equations formed on this assumption from all available material, this first approximation served to correct each Δ to the centre of the small square to which it was assigned, and an amended solution for the values of the corrections at the square centres was obtained.

A similar solution was also carried out for the measured diameters as well as for the measures of diffraction displacements. The corrections to the diameters proved to be small, and the law of their variation over the plate appeared to be poorly determined from the amount of material available. For our present purpose it is sufficient to adopt the measured diameters without any correction to the plate centre.

The stars available for the determination of these corrections were divided into four groups according to the mean of all measured diameters of each star. As mentioned in § 4, the group of the brightest stars has been excluded. The following data relate to the groups retained:—

Group	(i)	(ii)	(iii)
Mean of all measures of diameter for any star	0·100 to 0·220	0·220 to 0·270	0·270 to 0·340
Number of stars	41	84	72
Number of equations of condition	82	159	154
Mean diameter of group	0·191	0·245	0·300

After the completion of the first solution, in order to smooth inequalities due to the low weight of the individual determinations, the weighted mean of corrections in each set of four adjacent squares was formed and assigned to the common corner. It was found that the corrections to reduce measured Δ's to the plate centre are well represented by a law of the form

$$\text{Correction} = f(x) + F(y).$$

For the adjustment of each Δ to its square centre the first solutions for the three groups were combined, double weight being assigned to group (ii) as the P.E. of unit weight in group (iii) was found to be about $\sqrt{2}$ times that for either (i) or (ii). The results of the final solutions for the corrections to be applied to Δ are given in Table I.

Table II gives the derived values of $f(x)$ and $F(y)$ before the application of the constant to give zero correction at the centre. The corrections to the centre are plotted in fig. 1, in which the vertical strokes extend $2r$ above and below the point given by the solution, r being the probable error. There is strong evidence that $f(x)$ depends on the magnitude, whereas $F(y)$ does not. The three determinations of $F(y)$ have been combined with weights 1, 2, 1 respectively, and the graphs give the adopted corrections from which tables with arguments (x, D) and y were constructed, D being the diameter of the central image.

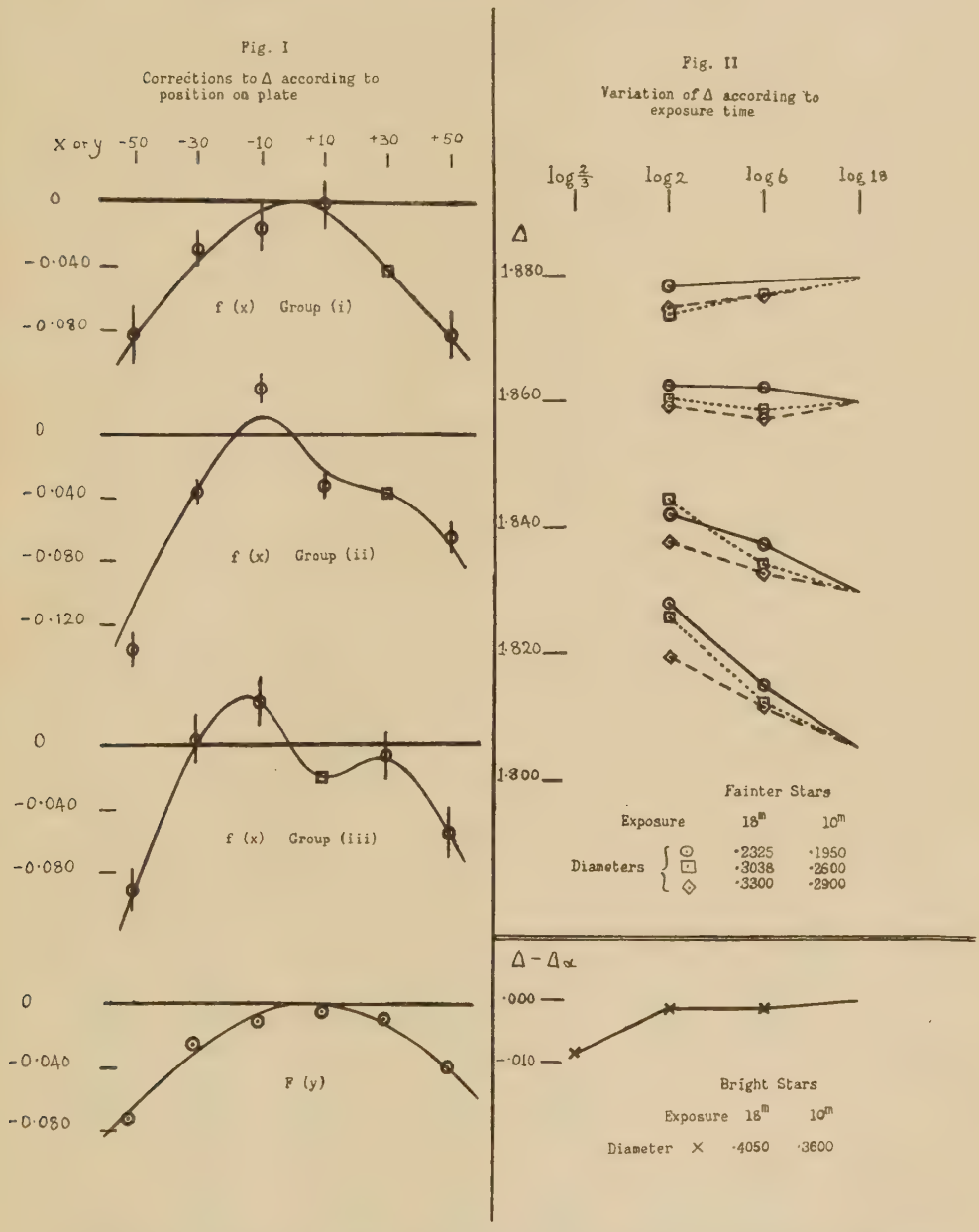
Each value of $f(x)$ is in general the weighted mean of six differences of pairs of terms in Table I with equal y's, the weight of a difference being $\frac{w'w''}{w'+w''}$. After the correction of each entry in Table I to the same x as the correction arbitrarily put zero, each value of $F(y)$ is also given by the weighted mean of six determinations. A high arbitrary weight was assigned to the adopted zero value. $F(y)$ could be obtained by the same procedure as $f(x)$. There is a slight systematic difference between the results of the two methods, due to the fact that in the first an error in an adopted value of $f(x)$ will affect each $F(y)$. Such a difference does not affect our final table of corrections.

The differences between the individual determinations of $f(x)$ and $F(y)$ and the weighted means furnish the probable error of unit weight in our solution. These were derived for Δ_1 and Δ_2 also. There is good agreement between the results from $f(x)$ and $F(y)$, and the following are adopted:—

		Probable Error of Unit Weight in Solution for Corrections to		
		Δ	Δ_1	Δ_2
Group	(i)	·016	·008	·012
	(ii)	·017	·010	·010
	(iii)	·025	·015	·013

Since unit weight here corresponds to a difference between two determinations of Δ, the probable error of a single determination of Δ, including night to night variations due to changed conditions of the atmosphere, instrument and development, is 0·011, corresponding to 25 A. The probable error of a single determination of Δ was also derived from a random selection of stars

by comparing the results of different nights after the application of these corrections. Three groups were considered according as both, one only, or



neither of the pair of images had both co-ordinates numerically less than 45′. The same result was obtained from each, showing that Δ is determined with equal accuracy over the whole of the measurable area of the plate (maximum of 60′ in either co-ordinate). The value found for r is 0·019 corresponding to 43 A. The excess of this over the previous determination

TABLE I

Solution for Corrections depending on Position on Plate

Co-ordinates		Unit of Correction 0·001 rev. Magnitude Group (i)		(ii)		(iii)	
x	y	Wt.	Corrn.	Wt.	Corrn.	Wt.	Corrn.
−50	−50	·78	− 83·1	1·54	−137·8	6·54	−133·5
	−30	·92	− 76·1	2·32	−110·4	3·29	− 68·4
	−10			4·10	− 51·7	1·95	− 80·3
	+10	·91	+ 13·9	1·41	−109·4	5·62	− 44·7
	+30	·67	− 53·8	1·64	−104·0	2·71	− 38·4
	+50	1·04	−107·6	2·93	−112·3	1·69	−146·9
−30	−50	1·49	− 83·2	2·07	− 66·9	2·42	− 29·2
	−30	5·40	+ 23·6	3·57	− 4·6	2·42	+ 17·6
	−10	1·18	− 28·6	8·70	+ 32·7	1·58	+ 40·1
	+10	1·78	− 8·5	1·80	− 3·0	2·25	+ 40·3
	+30	4·10	+ 12·2	3·20	+ 28·8	1·91	+ 38·1
	+50	1·07	− 72·9	7·75	− 6·0	1·63	+ 8·9
−10	−50	1·31	− 3·4	3·56	− 7·2	3·61	+ 0·1
	−30	2·30	+ 7·3	2·65	+ 61·9	2·67	+ 24·7
	−10	1·06	+ 8·9	4·97	+ 85·6	1·64	+ 52·9
	+10	1·22	+ 42·4	3·17	+ 96·1	1·86	+ 83·8
	+30	1·00	− 9·9	2·33	+ 67·2	1·76	+ 51·0
	+50	1·20	+ 5·9	3·91	+ 65·0	1·28	+ 44·5
+10	−50	·90	− 8·2	1·20	− 23·7	6·67	− 53·4
	−30	1·16	+ 29·0	2·13	− 2·2	2·16	+ 52·4
	−10	·78	+ 16·6	3·89	+ 24·8	1·93	− 8·8
	+10	·97	+ 51·0	2·00	+ 2·5	*[4·8]	0
	+30	2·00	+ 39·4	2·94	+ 7·4	2·95	+ 28·0
	+50	·75	− 5·0	4·48	+ 10·6	2·52	− 15·9
+30	−50	·49	− 42·4	2·67	− 81·2	2·34	− 65·5
	−30	4·24	− 32·5	4·10	− 13·5	1·66	+ 40·9
	−10	·98	− 14·1	3·89	+ 21·5	1·72	+ 31·6
	+10	1·74	+ 11·8	1·89	+ 15·0	3·01	+ 30·2
	+30	*[3·0]	0	4·24	+ 24·5	2·23	+ 37·9
	+50	1·21	− 45·3	*[6·0]	0	2·04	− 2·9
+50	−50	·68	− 55·7	3·34	− 63·0	2·28	− 75·7
	−30	1·13	− 63·4	3·86	− 35·9	1·98	− 80·2
	−10	1·04	− 99·8	3·08	− 20·1	1·17	− 12·7
	+10	1·04	+ 5·6	4·13	− 21·9	2·67	− 14·7
	+30	1·70	− 42·2	1·58	− 18·7	2·21	− 19·2
	+50	1·03	−108·2	4·17	− 35·7	1·52	− 37·6

* Arbitrary weights assigned in the formation of $F(y)$.

TABLE II

x or y	Magnitude Group (i) $f(x)$	(i) $F(y)$	(ii) $f(x)$	(ii) $F(y)$	(iii) $f(x)$	(iii) $F(y)$
−50	−39·6	−48·5	−98·8	−58·3	−71·1	−55·9
−30	+14·4	−16·9	+ 0·6	− 8·6	+24·0	+ 2·3
−10	+27·4	−34·0	+66·1	+25·7	+47·7	+ 5·2
+10	+42·6	+15·8	+ 4·9	+ 8·4	0	+16·4
+30	0	− 5·2	0	+13·8	+13·8	+ 8·8
+50	−40·1	−55·9	−28·4	− 3·2	−35·4	−28·1

is doubtless to be accounted for partly by errors in the system of corrections $f(x)$, $F(y)$.

The probable error for a single night is thus definitely larger for the Cape Astrographic refractor than for the Greenwich 30-inch reflector (stopped to 20 inch), as found by Davidson and Martin.* On the other hand, the area available for measurement without loss of accuracy as compared with the centre of the plate appears to be the same in both cases.†

6. *Reduction to the Zenith due to the Effect of Differential Atmospheric Absorption.*—The corrections required to reduce observations at great zenith distances to the zenith can conveniently be obtained by differential measurement, by exposing a plate on a field at a large eastern hour angle and exposing the same plate later to the same field near the zenith. Both exposures are then subject to the same conditions of development. The result may be vitiated by a general change in atmospheric absorption and a change in the sensitivity of the plate in the interval between the exposures. An error of this type is readily eliminated by exposing a second plate on an area near the zenith at about the same time as the first area is observed at a large eastern hour angle, and later on the same area at a large western hour angle at about the time the first field is observed near the zenith. The mean of the results of two such plates will be free from the effects of such changes.

The observing scheme actually adopted involved exposing one plate successively on two areas rising from the east, and another plate successively on two areas setting in the west. Each area was photographed at four altitudes, giving the following scheme of images :—

α, near the zenith;
β, γ, δ, at increasing zenith distances down to 62°.

The programme was carried through on two nights with exposure times according to the scheme of the main observing programme, increased exposure times being allowed for photographing areas at low altitudes.

For all stars measurable on these plates differences $(\beta - \alpha)$, $(\gamma - \alpha)$, $(\delta - \alpha)$ have been formed from the Δ measures, and the small corrections necessary

* *M.N.*, **82**, 65, 1921. † *Ibid.*, **84**, 425, 1924.

to reduce each Δ_β, Δ_γ, Δ_δ to the position of the corresponding α image on the plate have been applied. Table III gives the means of all material in each of five overlapping groups, according to the value of Δ at the zenith (corrected to plate centre), with their probable errors and the number of stars contributing to each mean. In fitting these differences to a law of the form

$$d\Delta = A(\sec z - 1)$$

it is clear that the differences for images β and γ are systematically too large for the bluer stars but not for the redder stars, the effect being most marked

TABLE III

Increase of Δ with increasing Zenith Distance

Exposures. Mean Z.D. Sec. z – sec. 5°		α 5°	β 27° ·118	γ 51°·5 ·603	δ 62° 1·126		$\frac{d\Delta}{\sec z - 1}$	
Range of Δ_α	Mean Δ_α and λ_{eff}	Mean Excess of Δ over Δ_α and P.E. of mean. Unit 0·0001 rev.				No. of Stars		Expressed in A
⩽ 1·770	1·753 3986 A.		180 ± 40	300 ± 40	360 ± 40	8	320	+73 ± 8
< 1·800	1·777 4041 A.		112 ± 13	179 ± 13	265 ± 13	85 67	236	+53 ± 2·6
1·790 to 1·830	1·812 4120 A.		66 ± 9	113 ± 9	172 ± 12	151 103	153	+35 ± 2·4
>1·820	1·843 4191 A.		22 ± 10	82 ± 10	156 ± 13	118 85	139	+31 ± 2·6
⩾ 1·840	1·854 4216 A.		0 ± 10	60 ± 30	150 ± 35	15 11	130	+30 ± 7

for β. This discordance has been attributed to variations in atmospheric absorption and instrumental conditions during the course of the night, as the exposures for β and γ were not approximately simultaneous with exposures at the zenith. These variations during a single night appear to be comparable in order of magnitude with the changes from night to night. The coefficients A in the last column of Table III have therefore been derived from the $\delta - \alpha$ differences only. It is clearly shown that reddening due to atmospheric absorption is almost independent of wave-length for red stars, but increases rapidly with decreasing effective wave-length for blue stars. By grouping the observations according to the diameters of the central image it was found that the amount of the apparent reddening does not depend on apparent magnitude.

Hertzsprung has investigated the effect of atmospheric absorption on observations of effective wave-lengths by means of a coarse wire grating,* and found

$$\Delta\lambda/\Delta\text{atm.} = (+35 \pm 4)\text{A./atm.}$$

In another publication he has examined the effect at two different effective wave-lengths giving 37 A. for $\lambda_{eff} = 4250$ and 26 A. for $\lambda_{eff} = 4500$.† Similar effects have been found by Lindblad.‡

7. *Corrections to Homogeneous System when Exposure Time is Varied.*— Davidson and Martin found that an increase of exposure first displaced the images of the first order towards the red, and with further increases of exposure this displacement towards the red reached a maximum and then decreased. Relative displacements between 10^m and 2^m exposures were fitted to a general quadratic in λ and m in order to construct a table of the magnitude equation for their reduction to standard density.§

In order that grating plates of Eros with exposure times differing from 10^m might be reduced to the system with the standard exposure time, four plates were taken each with exposures of 18^m, 6^m, 2^m and 30^s. Small differential corrections have been applied to the shorter exposures to reduce to the position of the corresponding 18^m image on the plate, and the results examined in twenty overlapping groups according to the diameters and Δ of the 18^m exposure reduced to the centre of the plate.

Within the range of colour, magnitude and exposure time covered here, the available material showed no systematic variation according to colour in the increase of the measured diameter of the central image with exposure time.

The observations in the three bright groups have been fitted to a law of the form

$$\sqrt{D+h} = a + b \log t,$$

where t is the exposure time in minutes, *i.e.* a form analogous to Ross's extension of the Greenwich formula for magnitude, time replacing intensity.‖ As appears in Table IV, the values of h, a and b, giving the best fit, show a systematic variation with magnitude. Permissible large changes in h do not destroy this systematic variation, which remains particularly marked in a.

Table IV

Diameter of 18^m Exposure	h	$10^{\frac{3}{2}}\, a$	$10^{\frac{3}{2}}\, b \log 3$
0·560	0·300	+24·5	+1·82
0·390	0·120	+17·6	+1·90
0·320	0·030	+13·0	+2·16

If the diameter of an image was a function only of the expression arising either in Schwarzschild's or in Kron's law for photographic density, curves in which diameter is plotted against $\log t$ would differ from magnitude to

* *Publ. Potsdam Obs.*, Nr. 63, 1911.
† *Ast. Nach.*, **192**, 309, 1912.
‡ *Loc. cit.*
§ *M.N.*, **82**, 65, 1921.
‖ *The Physics of the Developed Photographic Image*, p. 93.

magnitude only in their displacement in the direction of the time axis. We use $D(t, D_T)$ for the diameter of an image formed with exposure time t when the same star gives an image of the diameter D_T in time T. Taking an exposure time t' such that a star of different magnitude gives

$$D(t, D_T) = D(t', D_T').$$

The departures from the above form due to the greater turbidity associated with a brighter source * are well represented by

$$D(nt, D_T) - D(nt', D_T') = k(D_T - D_T'),$$

in which

$$\left.\begin{array}{l} k = +0{\cdot}057 \text{ for } n = 3 \\ k = +0{\cdot}130 \text{ for } n = 9 \end{array}\right\} \text{ and } T = 18^{\text{m}}.$$

Though the amount of the available material is not great, the agreement between several determinations of k is very close over our range of magnitude and exposure time.

Variation of Δ with Exposure Time.—In the bright groups there is an initial increase of Δ with increased exposure. There is considerable discordance in the measurement of the dense images for the longer exposures, but there appears to be no further reddening. No systematic variation with colour is found, and the result is given for the mean of all effective wave-lengths. The fainter red stars show a slight increase of Δ with exposure time, but the fainter blue stars show a considerable decrease of Δ with increasing exposure, the change with colour being markedly progressive. The results are exhibited in fig. 2, double weight having been assigned in each magnitude group to the stars near the mean magnitude.

8. *Correlation of Δ with Spectral Type and Colour Index.*—The stars contained in the Henry Draper Catalogue and measured on the present series

TABLE V

Correlation of Δ with Spectral Type

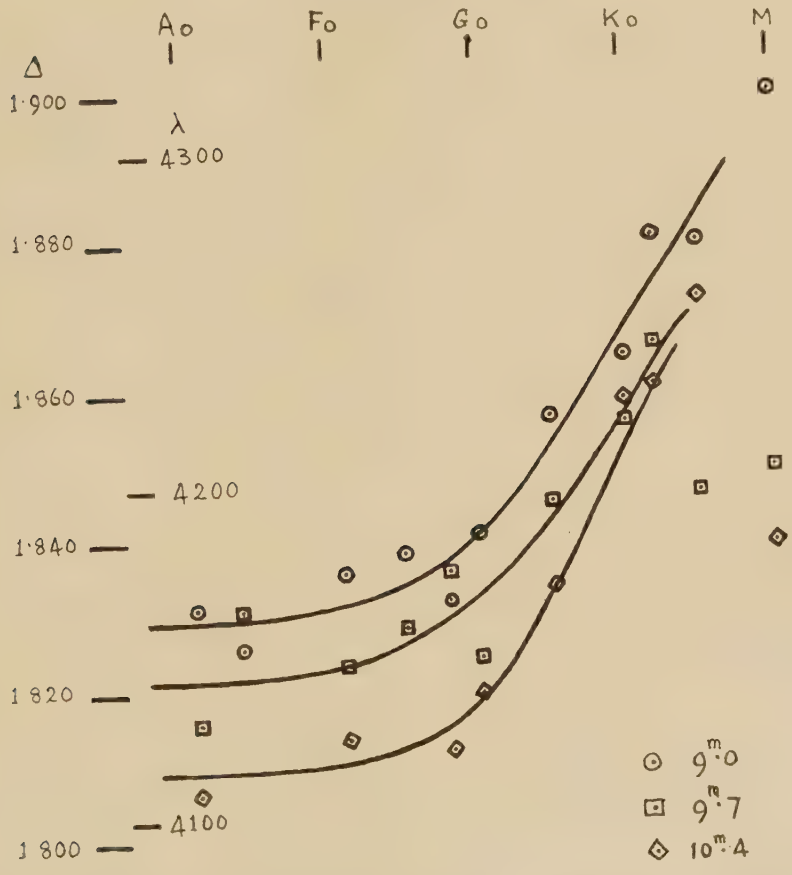
Mag.	9·00		9·70		10·40	
A0 and A2	1·8316	26	1·8151	30	1·8065	12
A3 and A5	1·8261	13	1·8312	16		
F0 and F2	1·8360	42	1·8238	18	1·8138	6
F5	1·8387	22	1·8288	32		
F8	1·8323	19	1·8362	25	1·8123	14
G0	1·8411	22	1·8247	30	1·8199	12
G5	1·8569	44	1·8454	43	1·8340	9
K0	1·8650	45	1·8560	65	1·8592	56
K2	1·8811	19	1·8665	38	1·8612	23
K5	1·8803	13	1·8465	15	1·8727	27
M	1·9005	4	1·8497	6	1·8396	10

* Ross, *loc. cit.*, p. 100.

of plates have been grouped according to spectral type and the photographic magnitude taken from the Henry Draper Catalogue. Table V gives the mean Δ in each group, reduced from the mean photographic magnitude of the group to magnitudes 9·00, 9·70 and 10·40 respectively, and the number of determinations of Δ contributing to the mean. Stars whose central

Fig. III

Correlation of Δ with Spectral Type



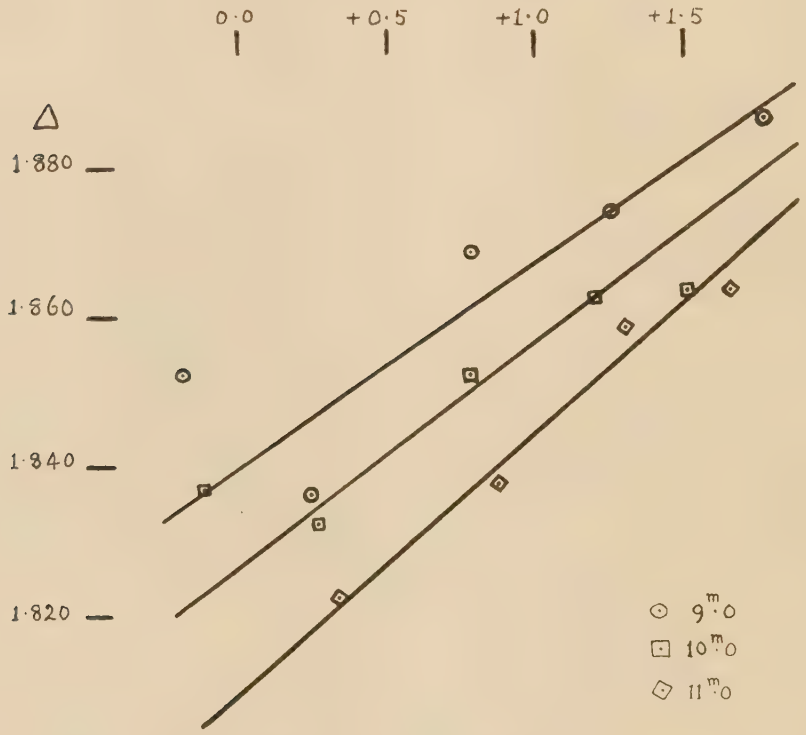
images have a diameter exceeding ·360 with exposure time 10^m have been excluded on account of the uncertainty of measurement and of the correction to the centre of the plate. The probable error of a single Δ from the groups in this correlation is 0·025, and is to be attributed partly to the presence of the magnitude equation as well as to the error of the night in Δ and accidental errors of classification.

The results are shown graphically in fig. 3 and contain a considerable magnitude equation, the increase in apparent effective wave-length for each increase of 1^m in brightness being 32·5 A. In view of the exclusion of the bright stars, which alone have been found in § 7 to show an apparent increase

in λ_{eff} with increased exposure time, a magnitude equation of this nature, which is confirmed later by a correlation with the colour indices of Seares, Sitterly and Joyner, may appear to be paradoxical. It should be noted, however, that the decrease in Δ as the exposure proceeds is more marked for the fainter stars, and the difference is greater for the bluer stars than for the red. This agrees with the nature of the correlation found with both type

Fig. IV

Correlation of Δ and Colour Index



and colour index. In the former case the redness of giants in their class will tend to reduce the amount of convergence of the correlation curves at the red end.

For each magnitude the relation to spectral type is similar to that found from the observations with the Greenwich reflector,* but the two series show a large difference in zero point (the Cape effective wave-lengths being low) and in scale, the scale of the Cape observations being only 0·4 that of the Greenwich observations. Types K5 and M do not give accordant results.

In a similar manner the correlation between Δ and the colour indices of the southern comparison stars for Eros determined by Seares, Sitterly and Joyner has been examined in three magnitude groups.† There was at least

* *Determinations of Effective Wave-lengths of Stars made at the Royal Observatory, Greenwich, in the Years* 1920 *to* 1925.

† *Ap. J.*, 72, 311, 1930.

one measure of Δ for almost all stars of this list, and many had three or four. The results are shown in fig. 4, together with the best linear relations between Δ and C for photographic magnitudes 9·0, 10·0, 11·0 on the system of the above paper. These linear relations can be satisfactorily represented by the relation

$$\Delta = +1{\cdot}9785 - 0{\cdot}0153m - (0{\cdot}0071 - 0{\cdot}0038m)C.$$

For the effective wave-lengths of tenth-magnitude stars this gives

$$\lambda = 4151 + 70C,$$

showing the low zero and small scale already remarked upon. Previous series of observations have shown it to be a general rule that effective wave-length determinations with refractors show a much smaller scale value than observations made with reflecting telescopes. On the basis of the probable errors of photographic and photovisual magnitudes from a single plate given by the above writers, Δ computed as above for an Eros reference star will have a probable error of the order 0·004, excluding the effect of the probable errors of the coefficients. The probable errors of a and b in the equations $\Delta = a + bC$ originally formed for each magnitude vary from 0·006 for faint stars to 0·003 for bright stars. The above equation can thus be satisfactorily employed to compute Δ from the magnitudes and colour indices of Seares, Sitterly and Joyner, and so provides a criterion by which comparison stars may be detected which are subject to large displacements through the action of differential atmospheric dispersion.

9. *Grating Plates of Eros ; Spectral Type and Colour Index.*—In Table VI are given the results of measures of six grating plates of Eros, having seventeen exposures in all, taken on the dates given in column two with the exposures shown in the first column. Column three gives the time interval between the mid-exposure time and the previous principal minimum. The measured diameters of the central images of the stars and of Eros were used to derive the photographic magnitude of Eros on the system of the Henry Draper Catalogue, and also of Seares, Sitterly and Joyner's magnitudes of Eros stars, the ephemeris magnitudes with a correction for light variation being used to smooth observing errors. The adopted magnitudes of Eros are given in column four.

On each plate all available stars were measured for which there were determinations of Δ from the series of plates already discussed (usually means from three or four plates) as well as Eros, the number of such stars being given in column one. The diameters and Δ have been reduced to standard exposure time by means of graphs prepared from material discussed in § 7, and each Δ was then corrected to the plate centre. The result for each star was compared with the value in the catalogue (prepared from all the material on the forty-five plates), from which the correlations with spectral type and colour index given in the last paragraph were derived. The resulting plate correction was applied to the value of Δ derived for Eros to give the final result contained in column five. The deduced spectral type and colour index (on the system of Seares, Sitterly and Joyner) are given in the remaining columns of the table.

TABLE VI

Grating Photographs of Eros

Plate Exposure. No. Stars	U.T. of Mid-exposure 1931	Interval from principal min.	Mag. H.D. Photog.*	Final Δ	Sp. Type	Colour Index
	d h m	h m				
†3362	Feb. 8 22 27	2 38	9·52	1·851	G7	+0·67
$4^m·5$						
11	32	2 43	9·50	1·865	K1	1·11
	23 7	3 18	9·23	1·861	G9	0·90
	11	3 22	9·22	1·862	G9	0·91
3403	Feb. 13 1 16	1 20	9·29	1·847	G4	0·41
3^m						
6	19	1 23	9·29	1·847	G4	0·41
	22	1 26	9·29	1·839	G1	0·13
3537	Feb. 25 21 46	4 13	9·68	1·886		1·90
3^m						
17	49	4 16	9·68	1·881		1·73
	54	4 21	9·68	1·890		2·05
3616	Mar. 10 20 25	3 59	10·13	1·855	K0	1·03
8^m						
22	33	4 7	10·14	1·860	K1	1·21
	41	4 15	10·15	1·858	K1	1·12
†3628	Mar. 11 20 24	1 37	10·16	1·840	G6	0·55
10^m						
15	35	1 49	10·16	1·832	G3	0·33
3693	Mar. 18 20 53	1 30	10·35	1·849	G9	0·92
10^m						
22	21 3	1 40	10·36	1·849	G9	0·94

The discordances between the determinations on different nights are large, and are perhaps to be accounted for partly by the rapid motion of Eros at the time of the earlier observations, whilst on March 11 cloud interfered with the observation. Also such factors as variation of focus may affect the system of corrections to the plate centre from night to night. On account of the magnitude of the corrections, we would expect a small change of this nature to have a not inconsiderable effect on the deduced plate correction.

Giving equal weight to the mean value of Δ on each night (except February 8 and March 10, which have been given weights $\frac{1}{2}$ and $1\frac{1}{2}$, having regard to the number of exposures and of stars for the standardisation), the mean of all material gives a colour index $+0^m·99$ and spectral type K0 corresponding

* To obtain magnitudes on the system of Seares, Sitterly and Joyner it is sufficient to add 0·07 to these magnitudes.

† *Notes.*—3362 Eros images smaller than those of comparison stars.
3628 Cloud passing. Eros images smaller than those of comparison stars.

to 1·856 for Δ with magnitude 9·90 (H.D.). Using only the last three nights when the motion was slower, the mean value of Δ is 1·849 and of the magnitude 10·21, giving for the colour index and spectral type $0^m.86$ and $G8\frac{1}{2}$ in close agreement with the value $0^m.85$ given by Delporte and Bourgeois, which is referred to the system of Ross and Zug's magnitudes.* Eros thus appears to be definitely redder than the mean of the southern primary reference stars whose mean colour index in the system to which the present observations have been referred is $+0^m.59$. On the other hand, the mean value of Δ for all the southern primary reference stars measured is 1·8515, and agrees closely with the value for Eros itself when a little fainter photographically than the tenth magnitude. On account of the magnitude equation in Δ visual selection of comparison stars tends to reduce in mean the variation of Δ with colour for photographic observations, since the selected blue stars tend to be brighter photographically than the red stars.

If, however, the assumption is valid that the variations of Δ are proportional to variations of refraction displacements for photographs taken under similar circumstances, the comparison should be made with the mean value of Δ for reference stars with an average distribution over the plate. Restricting the selection to stars for which neither co-ordinate exceeds 45 feet, this mean value exceeds the value at the centre by 0·030. The effect on the solar parallax from this cause opposes the effect due to the redness of Eros as compared with the reference stars, and is greater in magnitude.

Differential atmospheric dispersion will thus introduce systematic errors into the solar parallax derived from photographic observations with refractors in three ways:

(*a*) Through differences of colour between Eros and the reference stars.
(*b*) Through differences of magnitude between Eros and the reference stars.
(*c*) Through the positions of the reference stars on the plates.

Summary

On account of the differing refraction displacements for images of stars of different colour, due to atmospheric dispersion, systematic error in the solar parallax derived from observations of Eros will be avoided only if the mean of such displacements for comparison stars agrees with that for Eros. The distances Δ between the first-order images when star fields are photographed through a coarse wire grating of suitable dispersion, with a suitable uniform exposure time, afford criteria by which to make a selection of comparison stars which will eliminate such systematic error.

Measures of Δ for the southern comparison stars, and stars of the Henry Draper Catalogue in the same fields, have been reduced to a homogeneous system by applying corrections to reduce to the centre of the plate and to the zenith. The results have been expressed in terms of the correlation between the system of Δ and spectral type, and also in terms of the correlation with

* *Bull. Ast. de l'Obs. Royal de Belgique*, I, No. 6.

the colour indices of the southern Eros stars given by Seares, Sitterly and Joyner, a division into three magnitude groups being made in each case.

Corrections to be applied when the exposure time differs from the adopted standard have been determined and used to correct grating plates of Eros with various exposure times. The final reduction to our system of Δ was effected by a comparison of Δ for the stars on these plates with the values already determined. Eros is found to be redder than the mean of the southern comparison stars, but the mean value of Δ for these stars agrees with Δ for Eros early in March, when it was photographically rather fainter than 10^m.

For any plate of the solar parallax series the apparent parallactic displacement will be

$$p_E + \kappa\left(\Delta_E - \frac{1}{n}\Sigma\Delta_S\right)\tan z,$$

where p_E is the real parallactic displacement of Eros, and Δ_E is derived by means of the formula of § 8 from the magnitude of Eros at the time of the observation and its colour index. Δ_E and Δ_S are corrected to the zenith distance of the observation, $\Sigma\Delta_S$ applying to the reference stars according to their position on the plate. If the second term could be kept small only by the exclusion of a large number of stars, it would be preferable to retain all the stars and include the constant κ as an unknown.

The small scale value in the relation of effective wave-length to colour index for refractors is favourable to a determination of the solar parallax in which systematic error due to colour remains small. It is noted, however, that systematic errors due to the magnitudes of the comparison stars and to their positions on the plate will also arise from the effect of atmospheric dispersion, and will be comparable in importance with errors due to colour. Apart from the latter source of error, which for refractors appears to be considerable, the nature of the relation between Δ and colour index and magnitude suggests that a term

$$\{\lambda(m_E - \overline{m}) + \mu(c_E - \overline{c}) + \nu(m_E c_E - \overline{mc})\}\tan z$$

might be adopted as representing the effect of atmospheric dispersion, in which λ, μ, ν are constants depending on the instrument.

Royal Observatory,
Cape of Good Hope :
1933 *February* 10.

THE VARIABLE SPECTRUM OF γ CASSIOPEIÆ.

William J. S. Lockyer, M.A., Ph.D., Major (late R.A.F.).

The spectrum of this bright star (H.D. 5394, 1900 R.A. $0^h\ 50^m{\cdot}7$, Dec. − 60° 11′, Ptg. Mag. 2·01) has probably been photographed many hundreds of times on account of the interesting spectrum which it displays. The star has been assigned a visual magnitude of 2·25, and, so far as is known, no

evidence has been given to demonstrate any light variation, or any change in the photographic spectrum.

This spectrum is of the Bo-type, and its full notation, according to the International Astronomical Union,* would be Boemnτ, as it contains bright metallic lines, nebulous lines, and bright hydrogen lines which extend down to $H\tau$ in several photographs taken at the Norman Lockyer Observatory.

The chief characteristics of the spectrum are as follows :—

Each of the hydrogen lines consists of two bright lines, with a sharp absorption line between them, situated centrally over a broad absorption band. These bright pairs of lines are very strong at $H\beta$, but diminish rapidly in brightness and become wider apart, owing to prismatic dispersion, as the violet is approached. Numerous distinct double and single sharp bright lines of the ionised metals are visible, and also broad nebulous helium absorption lines. A great amount of continuous spectrum is also present (see Plate 9).

The object of the present communication is to show that the spectrum of this star is not constant but variable, for it is found that the relative intensities of the lines forming the bright components of each of the hydrogen lines $H\beta$, $H\gamma$, $H\delta$, $H\epsilon$, $H\zeta$ and $H\eta$ distinctly change, and in a manner which suggests a probable period which may be of the order of about four years.

No reference is made in this paper to the behaviour of any of the other lines in the spectrum, although distinct changes of intensity have been observed in some of them in the Sidmouth spectra.

Before proceeding to describe the observations which have led to this conclusion, it would be appropriate to sum up briefly the work of other observers who have attempted, but not succeeded, in detecting any changes in the spectrum.

The early observers who visually examined this star spectroscopically, namely, Secchi 1866, Huggins 1867, Vogel 1872, von Konkoly 1874–1883, von Gothard 1881–1883, and many others, did not agree as to the constancy of the spectrum. The chief variations recorded related to the relative intensities of the hydrogen lines. Sometimes $H\alpha$ was not seen while $H\beta$ was very distinct, and at other times $H\alpha$ was brilliant and $H\beta$ faint. Other lines, such as D_3, etc., were recorded as being present or absent on different dates.

Later observers, who employed photography, seemed to be very sceptical about the value of these early observations.

Thus Sir Norman Lockyer,† who discussed his photographs (59 in number) taken between 1888 November 20 and 1894 November 16, wrote:

"The lines $H\beta$, $H\gamma$ and $H\delta$ are constantly seen, and $H\epsilon$ and $H\zeta$ appear when the photographic conditions have been good. This suggests that their apparent absence, noted by some observers in the period 1874 to 1883, was possibly due to imperfect conditions of observations"; and on p. 176 he states: "During the period covered by the photographs there is no evidence of any change in the intensities of the principal bright lines."

At a much later date (1914) F. E. Baxandall,‡ discussing the above and

* *Trans. I.A.U.*, 3, 252, 1928. † *Proc. R.S.*, 57, 175, 1894.

‡ *Solar Physics Committee Publication*, p. 37, 1914.

also many later photographs, wrote: "An intercomparison of the best of these shows that there are no definite changes in the spectrum on different dates."

Father Sidgreaves,* who examined 52 of his plates taken at Stonyhurst and covering the years 1891 to 1899, also did not find any changes in the hydrogen lines, for he says:

"The hydrogen spectrum has shown no sign of alteration during the eight years of observation, but in the other parts of the spectrum there is a possibility of changes . . . may be due only to atmospheric effects on the photographic definition."

Ralph H. Curtiss † has perhaps made the most detailed study of the spectrum of this star, which is incorporated in the volume to which reference is given here. He also includes in it a very full summary of the early visual observations.

Referring to these early observations he writes (p. 9): "It seems that we may conclude at once that, during the period from 1874 to 1884 and possibly until 1888, the variations of the hydrogen emission lines, *if real*, were of short duration, as Copeland has suggested. After 1883 (or 1888) there seems to be little chance that any appreciable variation existed. . . . On the other hand, the constancy of the photographic record must be given great weight."

In discussing the seventy-four Ann Arbor photographs, which were taken between 1911 August and 1914 December, Curtiss comes to the conclusion (p. 30) that:

"There has been found no evidence of real variation from plate to plate or from year to year in the intensity of the emission lines of $H\delta$, $H\gamma$, $H\beta$, D_3, $H\alpha$ and of the metallic emission lines, nor in the central absorption of $H\epsilon$, $H\delta$, $H\gamma$, $H\beta$ and the metallic lines, nor in the dark $\lambda 4026$ and the "K" line. Estimates of the intensity relatively to the neighbouring continuous spectrum of the two parts of the emission components of $\lambda 4584$ and the hydrogen lines $H\zeta$, $H\epsilon$, $H\delta$, $H\gamma$ and $H\beta$ furnish no clear indication of variation in the relative brightness of these features of the spectrum of γ Cassiopeiæ. The indication is that constancy was the rule in this spectrum during the period of these observations."

Curtiss is thus led further to state that his spectrograms (p. 7) "bring out the close qualitative correspondence of the features of this spectrum in the years 1911 to 1915, with those of the photographic observations reported by Lockyer, Campbell, Miss Maury, Sidgreaves, Merrill and Baxandall, and further that they register no intensity in the spectral lines not accounted for by uncertainties of photographic contrast."

In a very recent paper on the spectrum of this star Dean B. McLaughlin,* who is investigating the Ann Arbor spectrograms taken by Curtiss, including others taken more recently, records a suspicion that there seems to be a slight change in the emission value of the ratio of the violet to the red component at the line $H\gamma$. He further states:

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* *M.N.*, 59, 508, 1899. † *Pub. Ast. Obs. Univ. Michigan*, **2**, 1, 1916.
‡ *Pub. Univ. Michigan*, 4, No. 3, 177, 1932.

"Especially in the interval 1915 to 1925 this ratio seems to slightly exceed unity on most of the plates. This behaviour does not appear to be shared by $H\delta$. Plates taken in 1926 and 1927 show $V/R = 1$, and later observations show the violet component slightly the weaker."

It will be gathered from the above references, excluding the visual observations, that the spectrum of this star has been considered to be of a constant nature, with the single exception of McLaughlin's statement above.

Referring now to my own observations, these extend from 1921 October 1 up to 1933 February 14, and consist of a series of 68 photographs. The first observations were published in 1924.*

This star was included in a number of bright-hydrogen-line stars which I was investigating. The purpose of this general inquiry was to determine, both by eye estimation and by means of a wedge micrometer, the difference in intensity between the violet and red components of each of the hydrogen lines in each star in order to determine any periodic changes. This I succeeded in doing in the cases of ϕ Persei † and H.D. 20336 ‡ (Camelopardalis).

In a second publication, entitled "Further Observations of Some Bright-hydrogen-line Stars," § I referred to ten more photographs of γ Cassiopeiæ which I secured between 1924 July and 1927 October. In describing these photographs I stated, with regard to $H\gamma$, $H\delta$ and $H\epsilon$, that "in every instance the bright components of each of the hydrogen lines are nearly of equal intensity, that of the violet being perhaps slightly the stronger. There is suspicion, however, in the last negative which was taken that the red components have slightly increased in intensity."

I further stated (p. 686):

"There is no record by any observers that this spectrum has ever undergone any change. It is quite possible, however, that there may be occurring small variations in the intensities or relative intensities of the bright components of the hydrogen which have escaped observation owing to the lack of sufficient dispersion in the spectroscopic apparatus employed. The dispersion of the Sidmouth instrument is quite sufficient to detect any such possible changes, and it is for this reason that this star has been and is being kept under observation."

It may be mentioned here that the Kensington Prismatic Camera, with which all the photographs have been taken, has an objective of 9 inches aperture, 120 inches focal length, with a 9-inch prism of 45° angle giving a dispersion of 46·0 mm. between $H\beta$ and $H\delta$.

The method of determining the differences between the intensities of the bright components of the hydrogen lines was to make an eye estimate with a magnifying glass ($\times 2$). Then it was estimated whether the violet component was *equal* to ($V = R$), a *little brighter* than ($V > R(L)$), *brighter* than ($V > R$), *much brighter* than ($V > > R$), and lastly *very much brighter* than ($V > > > R$), the red component.

A similar procedure was adopted as regards the red components in relation to the violet, namely, ($R = V$), ($R > V(L)$), etc.

* *M.N.*, 84, 412, 1924. † *Ibid.*, 85, 580, 1925; 86, 474, 1926.
‡ *Ibid.*, 86, 607, 1926; 91, 215, 1930. § *Ibid.*, 88, 685, 1928.

The differences were found for each of the five hydrogen lines $H\beta$, $H\gamma$, $H\delta$, $H\epsilon$ and $H\zeta$. In the case of $H\beta$ the bright components are so close together and the images so strong that the differences could only be detected when they were great: thus the values derived are not given in a subsequent table. While the components in the case of $H\gamma$ are well separated, the images are still so bright that small differences are still somewhat difficult to observe. Still greater separation occurs in $H\delta$, and more still in $H\epsilon$, and the components are not so strong, so better estimates can be made.

In the case of $H\zeta$ the estimates are perhaps a little less certain.

As all the four hydrogen lines have been treated quite independently in each negative, their very similar behaviour lends more weight to the changes in the differences of intensities recorded (see fig. 1) in the whole series of photographs.

In order to simplify the printing of the table of the estimates it was decided to adopt the following notation, replacing letters by figures, thus:

$R > V$	+ 5
$R > V(L)$	+ 2
$R = V$	0
$V > R(L)$	− 2
$V > R$	− 5
$V >> R$	− 8
$V >>> R$	−11

The reason why there are no equivalents for +8 and +11 is that these estimates do not occur.

In Table I, which follows, the letters "M" and "P" refer to the quality of the estimates made for each pair of the bright components, depending chiefly on the definition of the lines, these letters representing "moderate" and "poor" respectively. When no letter is given the estimate is considered "good."

As a check on the results of these estimated values, Mr. D. L. Edwards made a series of measures by means of the wedge micrometer, this method having been described by me in a former communication.*

The components of $H\gamma$ were selected for these measures and are given in Table I. In this case also "P" represent "poor" measures and blank spaces "good" measures.

Table I contains the detailed estimates and measures of each of the hydrogen lines mentioned above. The first column gives the number of the Sidmouth negative, the second and third the year, the month and day on which each photograph was taken. Then follow the eye-estimate values and quality for each of the bright components of $H\gamma$, $H\delta$, $H\epsilon$ and $H\zeta$, and finally the wedge measures and quality for $H\gamma$.

All the values given in Table I, with the exception of the first, which is omitted to shorten the diagram, are graphically shown in fig. 1. The first four series represent the plotted values of the eye estimates for each of the hydrogen lines $H\gamma$, $H\delta$, $H\epsilon$ and $H\zeta$.

* *M.N.*, 85, 581, 1925.

TABLE I

Neg. No.	Year	Date	Eye Estimates								Wedge Measures	
			Hγ		*Hδ*		*Hε*		*Hζ*		*Hγ*	
			E	Q	E	Q	E	Q	E	Q	M	Q
111	1921	Oct. 1	−2		−2?		−2	P	...		0·0	
356	1923	Sept. 11	−2		−2		−2	M	−5		+0·1	
412	,,	Dec. 20	−2		−2		...		...		−0·2	P
414	,,	,, 23	0		−2		−2	M	−2?	M	−0·1	
459	1924	July 17	−2		−2		...		...		+0·1?	
471	,,	Aug. 9	−2		...		...		...		0·0	P
508	,,	Nov. 12	−2		−5		−5	M	−2?	P	−0·1	P
521	,,	,, 19	−5		−2		−2	M	...		0·0	
529	,,	Dec. 3	−5		−5		−5	M	...		−0·1	
642	1925	Sept. 10	−5		−5		...		+2?	M	−0·2	P
656	,,	,, 23	−5		−2		−2		−2	M	+0·1	
856	1926	Sept. 20	0		−2		−2		0	M	0·0	
987	1927	Oct. 10	+2		+2		−2	P	...		−0·1	
1044	1928	Aug. 26	−2		−2		−2	M	−2		0·0	
1076	,,	Sept. 13	0		0		0?		0	M	0·0	
1100	,,	Oct. 17	0	M	−2	M	...		...		0·0	P
1108	,,	,, 30	−2?		−2		+2	M	−2	M	0·0	
1135	,,	Dec. 14	0		−2	M	−2	P	...		0·0	P
1146	1929	Jan. 24	−5		−5		−5	M	−2	M	−0·1	P
1162	,,	Feb. 12	−5		−5		−2		−5		0·0	
1180	,,	Mar. 27	−2	M	−5		−5	M	...		0·0	P
1198	,,	May 20	0		−5		...		...		+0·1	P
1209	,,	June 10	−2		−2		−8		0	M	+0·1	
1233	,,	July 29	−2		−2	M	...		...		−0·1	P
1239	,,	Aug. 12	−2		−2	M	−2	M	−2	M	0·0	
1252	,,	,, 29	−5		−2		−5	M	...		−0·1	
1269	,,	Sept. 13	−2		−2		−2	M	0		0·0	
1298	,,	Oct. 14	0		0		−5		−5		0·0	P
1323	,,	Nov. 13	0		−2		−5		−5	M	0·0	
1342	1930	Feb. 16	0		0	M	−2	M	−2	M	+0·1	
1404	,,	June 27	+2		0		−2	P	...		+0·2	P
1442	,,	Sept. 2	+2		+5		+2	P	+2	M	0·0	
1628	1931	Jun. 27	+2		+2		0	M	+2	M	+0·3	
1640	,,	July 28	+2		+2		0	M	...		+0·2	
1641	,,	,, 30	+2		0	M	0	M	−2?	M	+0·1	
1644	,,	Aug. 3	0		0		0	M	0		0·0	P
1654	,,	,, 23	+2		+2		+2		−2		+0·1	
1656	,,	,, 26	0		0		−2		+2	M	+0·2	
1657	,,	,, 26	+2	M	0		−2	M	...		+0·2	
1658	,,	,, 26	0		0		0	M	+2	M	0·0	
1663	,,	Sept. 6	0		+2		−2	M	+2	M	+0·3	
1673	,,	,, 9	+2		+2		0	M	+2		+0·3	
1677	,,	,, 22	+5		−2		+2		+2	P	+0·1	P

Neg. No.	Year	Date	Eye Estimates								Wedge Measures	
			$H\gamma$		$H\delta$		$H\epsilon$		$H\zeta$		$H\gamma$	
			E	Q	E	Q	E	Q	E	Q	M	Q
1692	1931	Oct. 14	+2		0		0	M	0		+0·4	
1707	,,	,, 28	0		− 2		0		0		+0·1	
1724	,,	Nov. 20	+2		+ 2		0		+ 2	M	+0·3	
1736	,,	Dec. 18	+2	M	0	M	+ 2	M	...		+0·2	P
1758	1932	Feb. 1	+2		+ 2		+ 2		− 2		+0·2	
1778	,,	Mar. 15	+2		− 2		...		+ 2		+0·1	
1782	,,	,, 18	+2		+ 2		+ 2	M	+ 2	P	+0·2	
1783	,,	,, 18	0		...		...		...		+0·1	P
1789	,,	May 8	0		− 5		− 8		− 2	M	0·0	P
1803	,,	June 13	0		0		− 8	P	...		0·0	
1812	,,	July 3	0		0		...		...		−0·1	P
1818	,,	,, 20	0		− 2		− 5		− 5		−0·1	
1825	,,	Aug. 12	−5		− 5		− 5	M	− 5	M	−0·1	
1836	,,	Sept. 27	−5		− 8		− 8		− 8		−0·2	
1845	,,	Oct. 10	−5		− 8		− 8		− 8		0·0	P
1855	,,	,, 28	−8		− 8		− 8		− 5		−0·2	
1875	,,	Dec. 3	−8		− 8		− 8		− 8		−0·1	
1886	,,	,, 23	−8		− 8		− 8		− 8		−0·2	P
1893	1933	Jan. 6	−8		− 8		−11		−11		−0·3	
1894	,,	,, 9	−8		−11		−11		−11		−0·3	
1896	,,	,, 9	−8		−11		−11		−11		−0·4	
1902	,,	,, 18	−8		−11		−11		−11		−0·2	
1912	,,	Feb. 11	−8		−11		−11		−11		−0·2	P
1916	,,	,, 12	−8		−11		−11		−11		−0·3	P
1920	,,	,, 14	−8		−11		−11		−11		−0·3	

The black dots represent the "good" estimates, the circles the "moderate" estimates, and the circles with a question mark the "poor" estimates, each of them being the mean value derived from the negatives taken during any one month.

The scales on the left show the original method of making the eye estimates, and those on the right the corresponding figures adopted. The scale of years is given at the top and bottom, subdivided into months, and each estimate is plotted exactly underneath each subdivision, the vertical lines representing the month of January of each year.

At the bottom of the figure are plotted the wedge measures for the line $H\gamma$, using the same time scale. The black dots represent the "good" and the circles the "poor" measures. The scale for the wedge differences is shown to the right and left.

The first fact to be gleaned from the plotted points of both the estimated and measured values is that a horizontal straight line drawn through them will not satisfy the observations. In other words, the relative intensities of the

bright components are not constant; it must be concluded, therefore, that the spectrum is variable and must be classified as such in future.

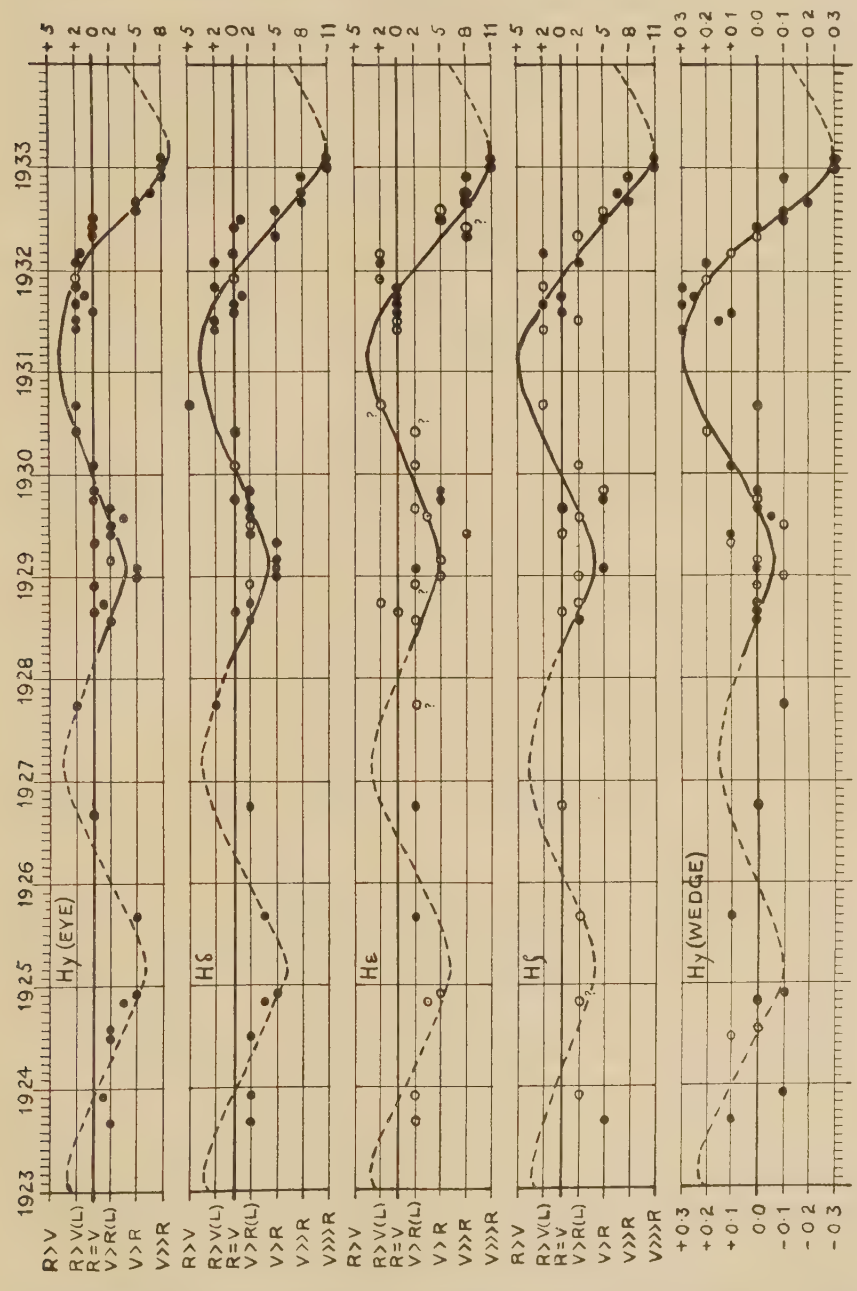


Fig. 1.

An attempt was first made to determine whether any regular variation was apparent, and so a mean curve was drawn to satisfy, as far as possible, the *Hγ* eye-estimate values.

Considering in the first instance the period between 1928 and 1932, both

years inclusive, in which the observations are most numerous, it seemed that a (shallow) minimum is suggested in 1929 March, a maximum in 1931 March, and a (deep) minimum in about 1933 March.

Continuing a smooth curve back to 1921, this curve satisfies the observations best with maxima in 1927 March and 1923 March, and minima in 1925 March and 1921 March.

(It should be mentioned here that the plotted points, representing the values derived from the Sidmouth photograph taken in 1921 October, are omitted from this figure in order to shorten the time scale of the diagram, and thus permit of a larger scale diagram.)

Thus there seems to be about a four years' interval between successive maxima or minima.

All the other curves, including that for the wedge measures of $H\gamma$, agree very closely with the curve for the $H\gamma$ eye estimates for the period from 1928 to 1932.

When continued back to the year 1921 they still corroborate in the main the first curve of the diagram.

It must be quite understood that this suggestion of a four-yearly variation is only of a very tentative nature, the observations prior to 1928 being so few, but it will be useful if only it draws attention to the importance, in the future, of taking photographs every month to secure a sufficiently long series of observations to determine exactly what variation is in progress.

An interesting feature to be noticed in the *wedge* curve is that the zero line lies midway between the maximum in 1931 and the minimum in 1933.

In 1930 I drew attention to the fact that the zero line in the case of H.D. 20336 * lay nearer the maximum than the minimum, while in the case of ϕ Persei † the opposite was the case.

It was considered quite worth while to inquire whether any variability could be deduced from observations of radial velocity measures published by numerous observers.

A variable velocity for this star was first announced by J. Hartmann ‡ in 1906 from observations made in 1900, 1901 and 1906. From his measures he stated: "Eine langsame Veränderung derselben erkennen lassen"; but regarding any definite period he said: "Genaueres uber die Natur der veränderung wird erst durch lange Zeit fortgesetzte zu ermitteln sein."

Since the above statement was published no other observer, so far as can be ascertained, has ever stated that the velocity is variable.

In order to test such observations to see whether the variation of about four years, suggested by the changes in the relative intensities of the bright components of the hydrogen lines, might be reflected in them, it was necessary to find out whether a sufficiently extensive series of radial velocity observations was made over a number of consecutive years.

Fortunately Ralph H. Curtiss's § observations are most opportune, because he seems to have made the most continuous series of radial velocity measures, his seventy measures on thirty-five different nights having been extended

* *M.N.*, 90, 224, 1930.
† *Ibid.*
‡ *A.N.*, 173, 102, 1906.
§ *Pub. Astr. Obs. Michigan*, 2, 26, 1916.

over the period 1911 July 17 to 1914 February 19. The velocities, as he states, "do not establish any changes from line to line or from year to year, but they do leave open the possibility of a slow variation with a period and amplitude similar in character to that apparently suspected by Hartmann."

He points out, however, that the observations now available are not sufficient to give γ Cassiopeiæ a place in catalogues of established binaries.

Curtiss's radial velocities are based upon the emission and central absorption components of $H\gamma$ and $H\delta$ and upon the emission components of $H\beta$, and the values given for each date are the means of all these values.

Fortunately, to supplement Curtiss's values, a few observations have been published by the Lick Observatory.* They were made in 1910 November, 1911 January, August and November, and in 1912 August. These values are based on the measurements of the two bright components of the $H\gamma$ line and the mean of each pair derived.

While the two systems of measurements are not strictly comparable, although each was considered the best way of determining the radial velocity of the star, both have been incorporated into one series here.

The following table brings together the available data :—

TABLE II

Date	Rad. Vel. km./sec.	No. of Obs.	Source
1910 Nov.	− 5·2	3	Lick
1911 Jan.	+ 2	1	,,
July	+ 1·1	1	Michigan
Aug.	− 9	1	Lick
,,	− 8·4	4	Michigan
Sept.	− 8·7	9	,,
Oct.	− 8·0	12	,,
Nov.	− 9·8	2	Lick
1912 Aug.	− 2·6	3	,,
Sept.	+ 0·4	1	Michigan
Nov.	−10·3	6	,,
Dec.	− 8·8	11	,,
1913 Jan.	− 7·1	3	,,
Feb.	−14·6	2	,,
Aug.	− 6·4	2	,,
Oct.	− 5·3	11	,,
Nov.	− 2·8	2	,,
Dec.	−12·1	2	,,
1914 Jan.	− 3·4	1	,,
Feb.	− 3·9	3	,,

The above values are graphically shown in fig. 2, the circles representing the Lick, and the filled-in circles the Michigan observations. The figures close to the circles represent the number of observations from which the mean value for the month has been derived. The horizontal broken line represents the mean of all the observations.

It will be noticed that in the year 1911 a cluster of values lies below this

* *Pub. L.O.*, **16**, 11, 1928.

mean line, while, at the end of 1913 and the beginning of 1914, another cluster lies above it. There is an undoubted suggestion of a change of value from 1911 to 1914.

An attempt was made to draw a smooth curve to represent the observations in a more satisfactory manner, and this is shown in fig. 2.

It should be stated that all radial velocity values for this star are nearly always negative, so that the prolongation of the curve in 1910 and 1914 should not go above the zero line. Thus it looks as if this curve should

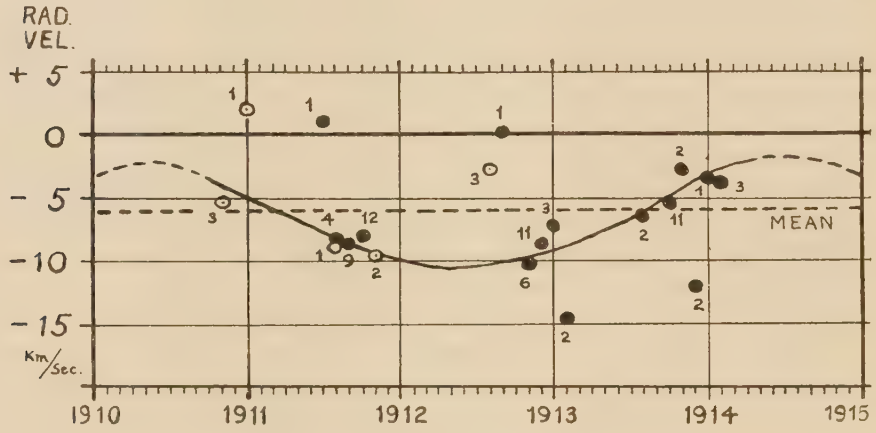


Fig. 2.

have minimum velocity values in 1910 and 1914, the maximum velocity being in 1912.

Thus the radial velocity values not only suggest a change in value but a variation of a possible four years' duration, similar in length to that indicated by the relative intensity changes of the bright components of the hydrogen lines.

It must be concluded, therefore, that there is now quite sufficient evidence to place γ Cassiopeiæ in the list of stars showing a variable spectrum, though many years must elapse before a definite period can be assigned to its variation.

Summary

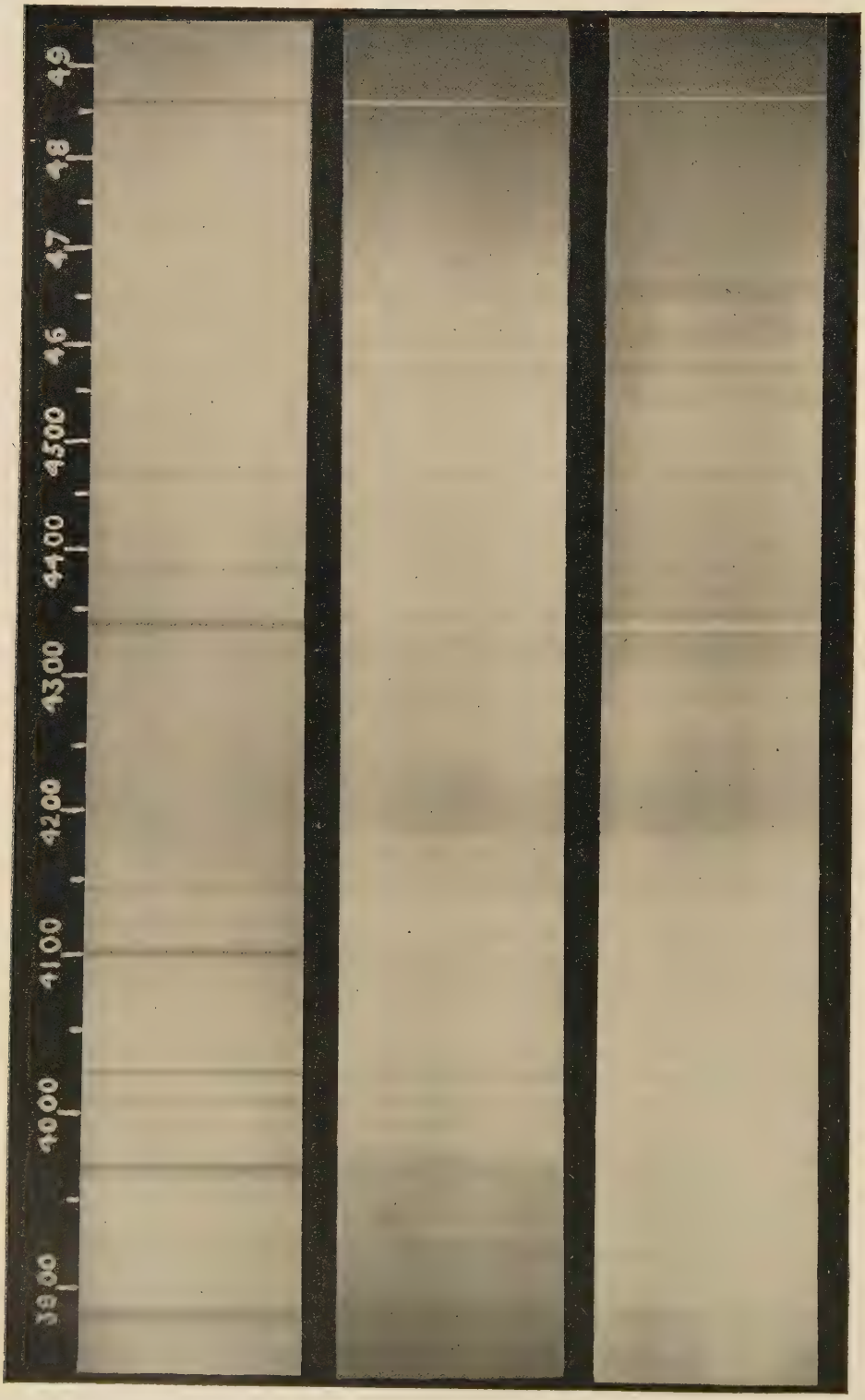
1. Since the spectrum of this star was first photographed every observer has upheld its continued constancy.

2. Suspected changes in the photographic spectrum were first announced in 1928 from photographs taken at the Norman Lockyer Observatory.

3. A discussion of the Sidmouth photographs taken between 1921 October 1 and 1933 February 14, sixty-eight in all, prove conclusively very marked changes in the spectrum as indicated by the variations in the relative intensities of the bright components of the hydrogen lines.

4. There is an indication that the changes may be of a periodic nature extending over about four years.

5. A brief discussion of some radial velocity measures indicates also a



Lockyer, Spectrum of γ Cassiopeiæ.

departure from constancy and suggests a periodicity of about the same length of time.

DESCRIPTION OF PLATE 9

This plate shows two spectra of γ Cassiopeiæ below with one of γ Orionis (Sp. Type B2) for comparison above. The upper spectrum of γ Cassiopeiæ was taken on 1931 September 6, when the violet and red components of the hydrogen lines were of about the same intensity. The lower one was taken on 1933 January 9, when the violet components were very much the stronger.

Attention may be drawn to the conspicuous broad absorption band which appears in both of the γ Cassiopeiæ reproductions between λ 4179 and λ 4239 ; this band is seen in the spectra of many B stars, and was described by me in a previous volume of this journal (**89**, 126, 1928), with reference also to this and other stars.

The above photographs were all broadened by the Clepsydra apparatus previously described (*M.N.*, **85**, 47, 1924).

ON THE IRREGULARITIES OF PERIOD OF S BOOTIS AND R CAMELOPARDALIS.

S. Plakidis, D.Sc.

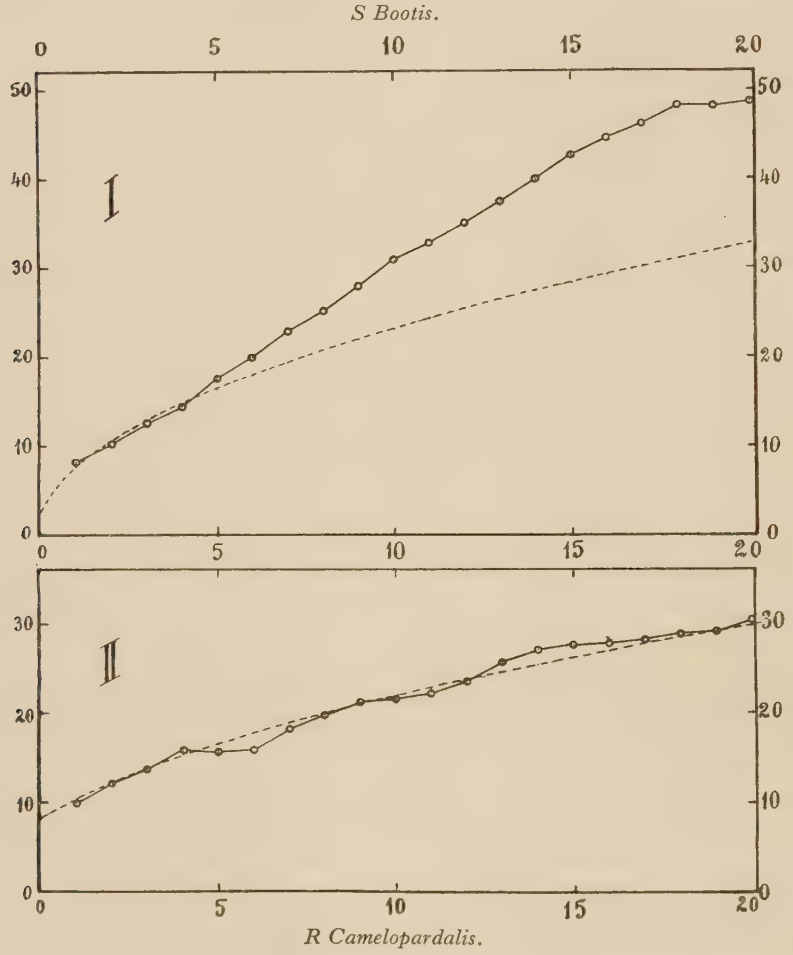
This paper is a continuation of an investigation on the irregularities of period of long-period variable stars. Twenty-one such stars have already been investigated on the basis of a new method of mathematical analysis and two more are now added to the list. An account of the method in question, as it was applied to o Ceti and χ Cygni, may be found in a previous paper (*M.N.*, **90**, 65, 1929) which was communicated in association with Professor Sir A. S. Eddington. A second paper summarising my conclusions as far as the remaining nineteen variable stars are concerned has appeared in *M.N.*, **92**, 460, 1932.

The two stars which are now under consideration are those discussed by Mr. W. F. H. Waterfield (*H.B.*, 880, 1930) in connection with changes of their periods. From the periodograms which he obtained by plotting the residuals of maxima against the epochs Mr. Waterfield draws the following conclusions :—

(*a*) In the case of S Bootis a change of period has taken place on two occasions, the first at about epoch 25, and the second at about epoch 60. Perhaps another such change is due in the near future, but this prediction will be proved or disproved by the next few maxima. At present it is beyond one's power to formulate any periodicity in the change of period of this star, but if any such exists the next change of period should take place at or about epoch 95.

(*b*) With regard to R Camelopardalis it seems evident from the plot that any attempt at present to explain the period changes of this star by a sine term is doomed to failure.

Having under consideration the classification of the possible types of irregularities of period, as it is given in the first of the above papers, I proceeded to determine the type of irregularity involved in the present case.



The results obtained by the application of the analysis mentioned above are summarised as follows :—

By means of the residuals (O − C) derived by Mr. Waterfield through a constant period of $269^d{\cdot}9$ and $271^d{\cdot}1$ for S Bootis and R Camelopardalis respectively I calculated the values of the $\bar{u}_x$ for these two stars. Plotting these values as ordinates against the values of $x = 1, 2, 3 \ldots 20$ as abscissæ I obtained the respective broken curves I and II. The smooth curves represent the best fit with the theoretical formula :

$$\bar{u}_x = \sqrt{(2a^2 + x\epsilon^2)},$$

in which the constants a and ϵ are :

For S Bootis	$a = 1{\cdot}90$	$\epsilon = 7{\cdot}28$,
,, R Camel.	$a = 5{\cdot}89$	$\epsilon = 6{\cdot}40$.

As it is evident from the close fit of the theoretical curves with the left-hand part of the respective curves of the observed $\bar{u}_x$, the irregularities of the periods of the stars in question follow the law of accidental and independent errors, being combinations of permanent with temporary irregularities.

This is proved also by the negative values of the correlation coefficients:

$$r_x = \frac{\overline{\epsilon_s \epsilon_{s+x}}}{\epsilon^2}$$

given in the following table:—

Values of r_x

r_x	S Boo.	R Cam.	r_x	S Boo.	R Cam.
r_1	−0·06	−0·23	r_{11}	+0·10	+0·02
r_2	+0·09	−0·04	r_{12}	−0·16	+0·05
r_3	−0·21	−0·07	r_{13}	+0·27	−0·11
r_4	+0·07	−0·05	r_{14}	−0·01	−0·02
r_5	+0·03	+0·01	r_{15}	+0·08	+0·06
r_6	−0·01	+0·14	r_{16}	−0·23	−0·22
r_7	+0·23	+0·04	r_{17}	−0·01	+0·02
r_8	+0·09	+0·01	r_{18}	−0·15	+0·01
r_9	+0·21	+0·04	r_{19}	+0·10	+0·03
r_{10}	−0·04	−0·08	r_{20}	+0·22	+0·04

For reasons already explained in the case of *o* Ceti I am inclined to think that the sinuosities of the $\bar{u}_x$ curve for R Camelopardalis are rather spurious and may be assigned either to gaps in the list of the observed maxima or to the accidental errors committed in fixing the date of maximum.

The above values of the constants a and ϵ are in favour of the conclusions arrived at in my last investigation, viz.:

(*a*) The flatness of the light-curve about maximum is here again to be held responsible for the high values of a. According to Müller and Hartwig,* as far as light-curves are concerned, S Bootis belongs to the most regular variables of the Mira type, while R Camelopardalis shows all the peculiarities of the stars of the same type, some of its maxima being sharp and some so flat that their determination is liable to considerable error. Moreover, some abnormal ascents of the light-curve of this star combined with the great difference between the various observers in fixing the maximum are to be considered as entailing the great deviation of the dates of the observed maxima from those derived through a constant period. All these features account very well for the stronger value of the constant a in the case of R Camelopardalis compared with S Bootis.

(*b*) Also for both these stars the ϵ terms are nearly in the same proportion as their periods, viz. the average deviation of an individual period of S Bootis from the mean is 2·7 per cent., while that of R Camelopardalis is 2·4 per cent.

* *Geschichte und Literatur der veränd. Sterne*, Bd. 1, 382 and 387.

The fact that there is no clear indication of the existence of some relation between the constant a and the period, or between the range of light fluctuation and the constants a and ϵ is again confirmed by the present case.

As a summary it may be added that the periodicity of the irregularity of period as shown by the periodigrams of Mr. Waterfield does not seem to be supported by our analysis. The whole irregularity of both stars consists rather of a purely accidental fluctuation of period combined with a purely accidental fluctuation of phase. In other words, both these stars may be classed among those of the normal pattern, the mean deviation from their mean period being less than 3 per cent., and the deviation of their phase being comprised within the limits of range already found for the 21 stars, viz. between one and eight days.

PRELIMINARY VALUES OF THE VARIATION OF LATITUDE AT GREENWICH IN 1932.

(Communicated by the Astronomer Royal)

This paper contains the preliminary values of the latitude variation derived from observations made with the Cookson floating telescope.

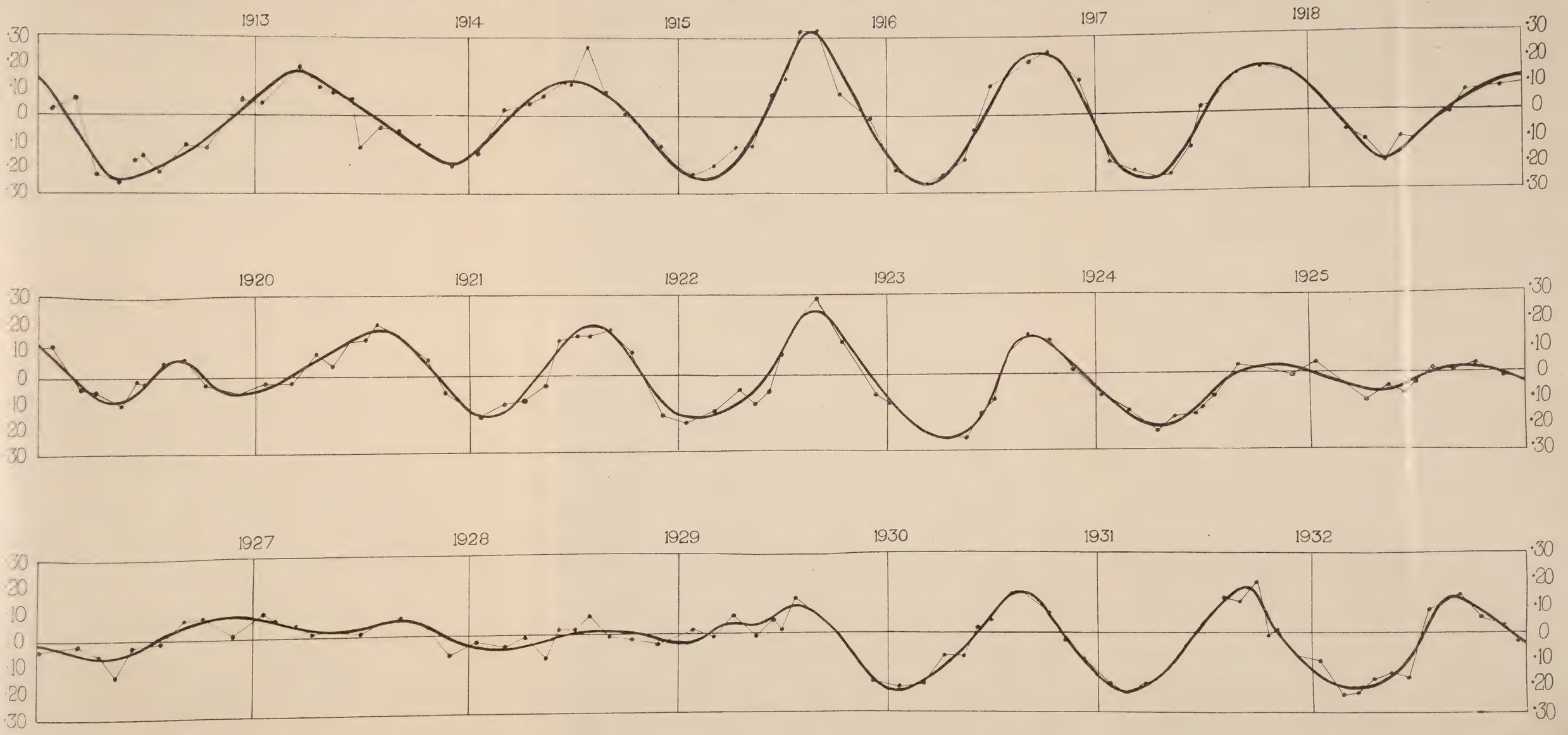
The star places used were those derived from the final discussion of the observations for the period 1919–1926. The number of plates obtained was above the average and they were fairly evenly distributed throughout the year.

The following table gives the value of the variation (mean latitude *minus* observed latitude) :—

Date	Variation	Number of Pairs
	″	
1932·036	+·101	80
·141	+·233	76
·207	+·226	78
·291	+·172	105
·367	+·159	89
·450	+·172	81
·548	−·097	93
·621	−·129	100
·686	−·159	96
·784	−·069	91
·881	−·033	82
·953	+·027	79
1933·033	+·034	100

From a curve drawn to represent the observed results as closely as possible, the following corrections, applicable to observed north polar distances, were derived :—

VARIATION OF LATITUDE AT GREENWICH



1932·0	+0″·12	1932·6	−0″·12
32·1	+0·19	32·7	−0·13
32·2	+0·21	32·8	−0·08
32·3	+0·19	32·9	−0·02
32·4	+0·15	1933·0	+0·04
1932·5	+0·05		

As this year completes a third series of seven years a diagram is given showing the variations throughout the entire period of observation with the Cookson telescope. A specially interesting feature of the diagram is the small range of variation from 1925 to 1929. The international observations indicate that the range during the same period in the second co-ordinate was also very small. It would seem that the natural period had been damped out and a fresh start was made in the second half of 1929.

Royal Observatory, Greenwich:
1933 *February.*

A STUDY OF THE EFFECT OF A THEORETICAL TIDAL VARIATION OF THE ZENITH ON THE ANNUAL VARIATION OF LATITUDE.

J. C. Dobbie, M.A., Ph.D.

Various attempts have been made from time to time to detect in astronomical observations the influence of the tidal variation of the vertical, and in a publication of considerable interest on the study of Earth tides from the variation of latitude * Stetson finds evidence of an effect amounting to as much as 0″·09. Such a value seems far too high to be due to a tidal variation of the vertical unless the tides are very much greater than existing theories allow. It seemed worth while therefore to investigate the type of variation that would be produced by the tidal attraction on the vertical, by treating a theoretical deflection computed for actual times of latitude observations in the same manner as the observations themselves. It is interesting to note that the greatest variation that can be produced with reasonable assumptions regarding the tides is of the order of 0″·01, and we should have to allow a very much greater Earth tide to account for the phenomenon that Stetson records.

The method followed in this investigation for determining the annual variation of latitude due to a theoretical tidal oscillation of the vertical is exactly similar to that employed at the Royal Observatory, Greenwich, for determining the observed annual variation of latitude. The Greenwich observations † for the years 1919–27 were chosen, and the observed residuals

* *Trans. Amer. Geophys. Union*, Eleventh Annual Meeting, 1930 May 1 and 2.

† Observations made with the Cookson Floating Zenith Telescope in the years 1919–27.

for the observations of each night were replaced by the value of a theoretical deflection of the vertical computed for the times of these observations.

In the Greenwich observations sixteen groups of stars were chosen, the groups being distributed in R.A. so that they could be observed successively throughout the year, any one group being observed at one time of the year in the evening and at another in the morning. The groups were each composed of four or five pairs of stars, the stars of each pair being chosen so that they should transit within a few minutes of one another, the one north and the other south of the zenith, the zenith distances being nearly equal. The zenith distance of each star was observed at time of transit and the mean zenith distance of each pair was computed, the mean being taken with proper sign, so that if the observed north and south zenith distances were equal the mean would be zero. The mean zenith distance of each star pair was calculated from the adopted star places, and the difference between the calculated and observed values formed the residuals. The corrected mean residuals of the star pairs for each group for every night on which the observations were taken are entered in the table at the end of the Greenwich volume, the residuals being taken in the sense, observed minus adopted zenith distance north. Ignoring the correction to the aberration constant, each mean residual gives an equation of condition of the form

$$\text{Mean residual} = z_p + t_q,$$

where z_p is the group correction to the group involved and t_q is the latitude variation appertaining to the period. Each period extends over about one-tenth of a year, and there are 78 periods in the whole seven years series.

In the present work the Greenwich Mean Residuals on the left-hand side of the above equations were replaced in every case by a theoretical deflection of the observer's zenith due to the action of the tides. These equations were then treated in a manner exactly similar to that employed by Greenwich, and the values for the z_p's and t_q's, being here the tidal contribution to the group correction and latitude variation, were computed.

Now since the Greenwich observations were taken with a telescope floating on a mercury surface, the tidal deflection which would be present at the time of observation would be due to the tilt of the mercury surface relative to axes fixed in a rigid earth. Unfortunately we know neither the amplitude nor the lag of this tide, but we do know the amplitude and lag of the tide in the mercury trough relative to the surface of the Earth. Michelson and Gale in their experiment on the tide in a pipe * found the amplitude to be 0·69 of that of the tide in an ocean covering a rigid earth and the lag to be 4°. From their experiment two solutions for the Earth and Ocean tides may be deduced,† one giving a negligible Earth tide. In this investigation this solution, giving a negligible Earth tide, is adopted, so that we are concerned, in fact, only with the Michelson tide.

* " The Rigidity of the Earth," *Astrophys. Journal*, **50**, 330, 1919.

† J. C. Dobbie, " The Ranges and Phase-displacement of the Earth and Ocean Tides," *M.N., Geophys. Suppl.*, 1930 January.

The form here assumed for the Ocean tide is taken as

$$\tfrac{1}{2}c[3\cos^2\theta - 1],$$

where c is the amplitude of the tide and θ the angular distance of the place of observation from the pole of the tide. It has been shown that on this assumption the Michelson deflection in a north-south direction may be written

$$\frac{1}{a}\frac{\partial}{\partial\lambda}0{\cdot}69\frac{Ma}{\frac{4}{3}\pi\rho R^3}\cdot\tfrac{1}{2}[3(\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos\overline{nt+L-4^\circ})^2-1],$$

where M is the Mass of the Sun or Moon,
a the radius of the Earth,
ρ the Earth's density,
R the distance of the Sun or Moon,
λ the latitude of the place of observation,
δ the declination of the Sun or Moon,
n the mean motion of the disturbing body,
t the interval after transit across some standard meridian, and
L the east longitude of the place of observation referred to this meridian.

This becomes

$$\alpha\sin 2\lambda[\tfrac{1}{4}-\tfrac{3}{4}\cos 2\delta+\sin 2\delta\cot 2\lambda\cos h-\tfrac{1}{4}(\cos 2\delta+1)\cos 2h]$$

or

$$\alpha(\sigma\sin 2\lambda),$$

where α is a constant.

Taking $\lambda = 51^\circ\ 28'\ 38''$ N. for Greenwich, σ becomes

$$0{\cdot}25-0{\cdot}75\cos 2\delta-0{\cdot}23003\cos h\sin 2\delta-0{\cdot}25(\cos 2\delta+1)\cos 2h.$$

The first step was to make a table of values of $\sigma\sin 2\lambda$ for all values of δ from $+25^\circ$ to -25° for each degree of declination and for all values of h from 0^h to 24^h for every four minutes of time. Actually it was only necessary to compute the values from 0^h to 6^h, the remainder being written down by symmetry. Next the values were multiplied by α and two separate tables were made, one for the Sun and the other for the Moon. Each value was computed to six decimal places. A displacement of sixteen minutes was made in the hour-angle columns to allow for the Michelson lag. It was next necessary to find the position of the Sun and Moon for each of the Greenwich observations, and these were found from the *Nautical Almanac*, the exact time of transit of the star groups being computed for each night. A table was compiled in a similar manner to the table at the end of the Greenwich volume by entering for the star pairs for each night, for all groups, the values of the zenithal displacement found from the tables for $\alpha(\sigma\sin 2\lambda)$. The values for each star pair for each night were meaned and weighted in a way similar to that employed by Greenwich, and a correction was applied to the mean of group in the case of incomplete observations.

From this point onwards the Greenwich procedure was followed exactly.

The resulting values for the theoretical z's and the theoretical t's are shown in Tables I and II.

TABLE I

$z_1 = +0''.0012$	$z_9 = -0''.0010$
$z_2 = +0''.0021$	$z_{10} = -0''.0022$
$z_3 = +0''.0022$	$z_{11} = -0''.0023$
$z_4 = +0''.0015$	$z_{12} = -0''.0025$
$z_5 = +0''.0024$	$z_{13} = -0''.0021$
$z_6 = +0''.0022$	$z_{14} = -0''.0017$
$z_7 = +0''.0005$	$z_{15} = -0''.0011$
$z_8 = -0''.0010$	$z_{16} = +0''.0017$

TABLE II

$t_4 = +0''.0079$	$t_{21} = 0''.0079$	$t_{38} = 0''.0074$	$t_{55} = 0''.0081$	$t_{72} = 0''.0083$
$t_5 = 0''.0107$	$t_{22} = 0''.0075$	$t_{39} = 0''.0051$	$t_{56} = 0''.0088$	$t_{73} = 0''.0068$
$t_6 = 0''.0102$	$t_{23} = 0''.0086$	$t_{40} = 0''.0036$	$t_{57} = 0''.0085$	$t_{74} = 0''.0077$
$t_7 = 0''.0088$	$t_{24} = 0''.0095$	$t_{41} = 0''.0064$	$t_{58} = 0''.0075$	$t_{75} = 0''.0079$
$t_8 = 0''.0058$	$t_{25} = 0''.0077$	$t_{42} = 0''.0074$	$t_{59} = 0''.0042$	$t_{76} = 0''.0074$
$t_9 = 0''.0041$	$t_{26} = 0''.0077$	$t_{43} = 0''.0097$	$t_{60} = 0''.0067$	$t_{77} = 0''.0065$
$t_{10} = 0''.0050$	$t_{27} = 0''.0072$	$t_{44} = 0''.0083$	$t_{61} = 0''.0067$	$t_{78} = 0''.0073$
$t_{11} = 0''.0066$	$t_{28} = 0''.0065$	$t_{45} = 0''.0079$	$t_{62} = 0''.0082$	$t_{79} = 0''.0047$
$t_{12} = 0''.0084$	$t_{29} = 0''.0036$	$t_{46} = 0''.0078$	$t_{63} = 0''.0076$	$t_{80} = 0''.0051$
$t_{13} = 0''.0081$	$t_{30} = 0''.0051$	$t_{47} = 0''.0069$	$t_{64} = 0''.0092$	$t_{81} = 0''.0106$
$t_{14} = 0''.0080$	$t_{31} = 0''.0065$	$t_{48} = 0''.0075$	$t_{65} = 0''.0060$	
$t_{15} = 0''.0077$	$t_{32} = 0''.0076$	$t_{49} = 0''.0057$	$t_{66} = 0''.0074$	
$t_{16} = 0''.0096$	$t_{33} = 0''.0087$	$t_{50} = 0''.0054$	$t_{67} = 0''.0086$	
$t_{17} = 0''.0091$	$t_{34} = 0''.0085$	$t_{51} = 0''.0069$	$t_{68} = 0''.0066$	
$t_{18} = 0''.0070$	$t_{35} = 0''.0096$	$t_{52} = 0''.0075$	$t_{69} = 0''.0042$	
$t_{19} = 0''.0053$	$t_{36} = 0''.0080$	$t_{53} = 0''.0107$	$t_{70} = 0''.0037$	
$t_{20} = 0''.0040$	$t_{37} = 0''.0081$	$t_{54} = 0''.0074$	$t_{71} = 0''.0073$	

Fig. 1 shows the theoretical annual solar tidal variation of the zenith for the hour angles 0, 3, 6 and 9 in a N.-S. direction, for the latitude of Greenwich, calculated from the expression $\alpha(\sigma \sin 2\lambda)$, where σ has the values already given and α the value $0''.0055$ for the Sun. The curves for hour angles 12, 15, 18 and 21 are practically the same as those for 0, 3, 6 and 9, being these curves displaced by half a year. For the Moon α has the value $0''.0118$, while the monthly variation due to the lunar tide for these hour angles of the Moon is otherwise very similar to the solar annual variation in fig. 1. There is of course a slow change of amplitude in this case with a period of nineteen years.

In the grouping and meaning of the values of zenithal variation for each individual observation of the star pairs the variation due to the Moon appears to be eliminated when an average is taken over the seven years so that the curve shown in fig. 2 (*a*), which gives the mean annual variation for these seven years before the group correction has been applied, resembles in character those of fig. 1. It is worth noting that fig. 2 (*a*) is very similar in character, though of amplitude only one-tenth, to curve (4) given on p. 30 of

the Greenwich results, which represents the residual Greenwich minus International results for 1923·0 to 1927·9.

Table I is represented graphically by fig. 2 (*b*). This gives the latitude variation after the application of the group correction. The mean value of the latitude variation is 0″·0073 north. This represents the constant

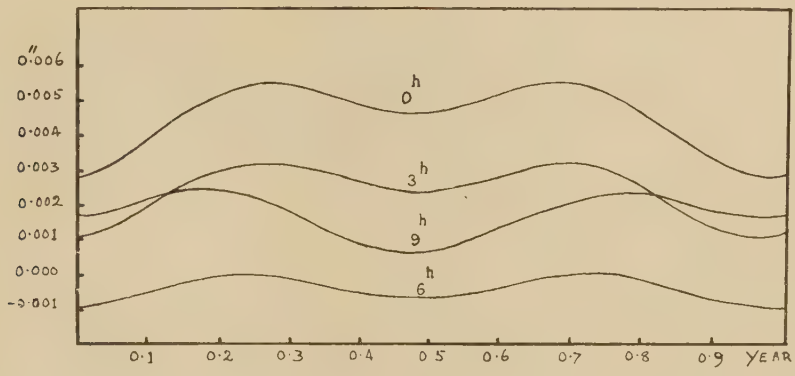


FIG. 1.—*The Theoretical Annual Solar Tidal Variation of Latitude for Hour Angles* 0^h, 3^h, 6^h *and* 9^h *calculated directly from the Formula* $a(\sigma \sin 2\lambda)$.

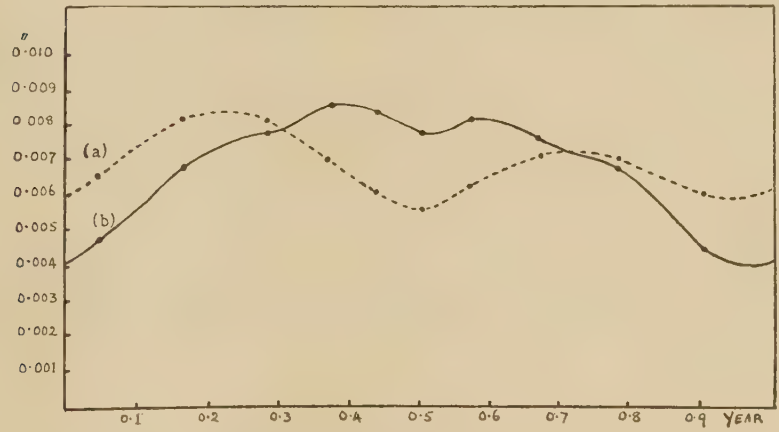


FIG. 2.—*The Mean Annual Theoretical Tidal Variation of Latitude Weighted for Times of Greenwich Observations.*

(*a*) *Before the application of the group correction.*
(*b*) *After the application of the group correction.*

contribution of the tides to the latitude of Greenwich, and a similar contribution must be present at all observatories according to their latitude. The effect may of course be greater than this if there is an Earth tide.

This constant is represented by the average value of the expression

$$a \sin 2\lambda[0{\cdot}25 - 0{\cdot}75 \cos 2\delta - 0{\cdot}23 \cos h \sin 2\delta - 0{\cdot}25(\cos 2\delta + 1) \cos 2h]$$

over the seven years.

In taking the average, the terms in sin 2δ and cos 2*h* disappear, leaving

$$a \sin 2\lambda(0{\cdot}25 - 0{\cdot}75 \overline{\cos 2\delta}),$$

where $\overline{\cos 2\delta}$ is the average value of cos 2δ over the seven years for the Sun and Moon.

This may be written

$$a \sin 2\lambda \left[0{\cdot}25 - \frac{0{\cdot}375[\sin 2\delta]_{-\delta}^{+\delta}}{[\delta]_{-\delta}^{+\delta}} \right].$$

Taking the limits of the Sun's declination to be $\pm 23\frac{1}{2}^\circ$, the value for the sun is $-0''{\cdot}0022$, while the value for the Moon, whose maximum declination during these years had an average value of about 20°, is $-0''{\cdot}0051$, making the total $-0''{\cdot}0073$. This is the deflection of the zenith in a southerly direction, so that the contribution to the latitude is $+0''{\cdot}0073$. This agrees exactly with the value found above from the mean of the *t*'s, thereby providing a check on the arithmetic.

It should be noted that if the latitude observations contain a tidal term of the nature discussed, part of the effect will, after reduction, appear in the group correction. This is due to the fact that, in observational work, the latitude is assumed constant throughout the night. This assumption is clearly the most reasonable one to have been adopted, since we do not know what law of nightly variation exists. The final theoretical value of the annual latitude variation given in Table II and meaned for the seven years in fig. 2 (*b*) shows little resemblance to the curve in fig. 2 (*a*), the real theoretical tidal variation of latitude. Thus it appears that the portion of nightly variation entering the observed group corrections in observational work may cause the true law of nightly variation to be obscured. It would appear, moreover, that the errors in the star places, though present in the group corrections, may be entangled with the variations of the night.

In conclusion I should like to express my thanks to Professor L. N. G. Filon, Dr. W. M. Smart and Mr. W. M. H. Greaves for their helpful suggestions and criticisms in the course of this work, and also to Dr. J. Jackson and Mr. H. W. Acton for their kind help in supplying data necessary for the reductions.

TRIAL OF A PROJECTION METHOD FOR MEASURING PHOTOGRAPHS.

Frank Schlesinger, Sc.D., and Arthur L. Bennett, Ph.D.

A good measuring engine may be said to possess three cardinal virtues: accuracy, rapidity, and convenience, the latter being understood to include a minimum of eye-strain and other forms of fatigue. It is only natural that in the early stages of this subject the first of these should have been given chief consideration; very seldom indeed, in the design and operation of engines, has a sacrifice of the least degree of accuracy been made in favour of any other advantages. This is more especially the case because the first methods of measuring photographs did not exhaust all the precision of which

the plates are capable. Gradually, however, methods have improved, until now, as is well known, we sometimes measure photographs with more precision than is necessary, the accidental errors of measurement being in many cases considerably less than those that are inherent in the plate and that have their origin in distortions of the film, magnitude error, atmospheric dispersion, and perhaps many other causes. In all likelihood the next steps that will be taken to enhance the accuracy of photography will lie in the direction of improvements in the plate itself. Under these circumstances we may well pause to devote more attention to the other virtues that engines may possess, particularly in the matter of convenience.

It is obvious that if we can successfully project the images to be measured upon a screen, together with the micrometer thread that is to bisect these images, we should at once relieve the observer of most of the eye-strain to which the ordinary micrometer microscope subjects him. Both eyes would then be used, and there would be no necessity for keeping the head and the body in fixed positions. A trial of this method in a tentative way appeared to us to hold out so much promise that we decided to go ahead with the experiment in more permanent form and to give it a thorough test. A search of the literature convinced us that no such test has ever been carried out, although the essential features of this plan must have occurred to many persons, as it did independently to both of us. We did find that E. Bäcklin,* of the department of physics of the University of Upsala, has measured laboratory spectrograms by what is essentially the same method as described in this paper, though there are many differences of detail. So far as we know, Mr Bäcklin has not made any numerical test of the accuracy of this method as compared with others. Nevertheless priority in actually constructing a measuring engine of this type belongs with him.

The measuring engine that we have modified for this experiment is of the long-screw type designed by Dr. Schlesinger in 1906 and described by him in the *Publications of the Allegheny Observatory*, **3**, 83. A dozen or more of these engines in somewhat smaller size have since been made by William Gaertner & Co. of Chicago; these are illustrated and briefly described in their Catalogue M, No. 1230. In all these engines the plate is moved to and from the observer by means of a rapid rack and pinion, while measurements are made with a microscope that is moved to the right or left directly by turning a long micrometer screw. If measurements are required in both co-ordinates, it is necessary to turn the plate through a right angle to obtain the second set.

The engine that we are using was constructed in our own shop early in 1931. It embodies several changes and improvements, the chief of which may be briefly mentioned before we describe the projection device. The micrometer head is a white celluloid strip, graduated with black divisions and figures, let into a light duralumin wheel of unusually large diameter, 22 cm. The circumference is divided into 1000 spaces, each of which is therefore about 0·7 mm. wide and corresponds to one micron on the plate since the pitch of the micrometer screw is 1 mm. Readings to the nearest

* *Review of Scientific Instruments*, **1**, 662, 1930.

micron can be made directly without estimation and without the use of a magnifying glass. This simple modification saves a good deal of eye-strain.

In the earlier examples of this engine a change-gear was provided to enable the micrometer screw to be turned at five times the direct speed and thus to permit the rapid examination of the plate. This has proved to be unnecessary ; with a finger hole in the face of the micrometer handle the observer can turn the screw three or four times a second, and this suffices for plates of moderate size. On this engine we have in fact installed a device of the very opposite character : a set of auxiliary gears enables the observer to turn the micrometer screw at one-fourth the direct speed. The best diameter for a micrometer handle is 8 or 9 cm. ; on the circumference of such a handle about one-quarter of a millimetre (or 0·01 inch) corresponds to a micron on the plate, and thus to set to this degree of accuracy requires somewhat delicate manipulation. With the reducing gear, however, a full millimetre on the circumference of its handle corresponds to a micron on the plate. With it settings are made more quickly and with more confidence, and the observer seldom spoils his reading by turning the screw too far, as he otherwise occasionally does. This is an improvement not only with regard to speed and convenience, but doubtless adds a little to the precision.

A disadvantage of the long-screw type of measuring engine, as compared with one in which a scale or a reseau is used, is that the measures are more sensitive to disturbances of the microscope due to manipulation or to slight impacts by the nose and other parts of the observer's face. A simple device for avoiding this difficulty (and one that we wish we had thought of earlier) is to separate the eyepiece from the tube of the microscope, mounting it upon a bridge of its own, parallel to the screw and capable of moving the whole length of the latter. The cross-wires remain, of course, in the microscope tube, at its very top. As the microscope moves to the right or left by means of the screw, the eyepiece must be moved along to follow it. This motion we might have made automatic by means of gearing with the micrometer screw, but we prefer to perform it by hand so as to avoid encumbering the micrometer setting with additional work. It is obvious that we may now adjust or change the eyepiece to any extent we please, since we cannot disturb in this way the relation between the cross-wires and the star image at the top of the tube, and it is only this relative position that counts in the measurement.

Most photographic plates are concave on the film side due to the contraction of the gelatine during drying. Our plates (21·5 by 16·5 cm.) are held in the measuring engine so as to bring the long edges of the film side into a fixed position, but on account of the concavity their centres may be appreciably below the plane of these edges. This depression may amount for our plates to 0·4 mm. and averages 0·16 mm.* This is enough to cause an appreciable lack of good focus. If we stop down the aperture of the

* We have recently measured 100 of our plates with a spherometer having a radius of 10 centimetres ; of these 2 are very slightly convex (by less than 0·03 mm.) and 12 are flat. All the rest are concave : 27 by 0·1 mm., 44 by 0·2 mm., 12 by 0·3 mm., and 3 by 0·4 mm.

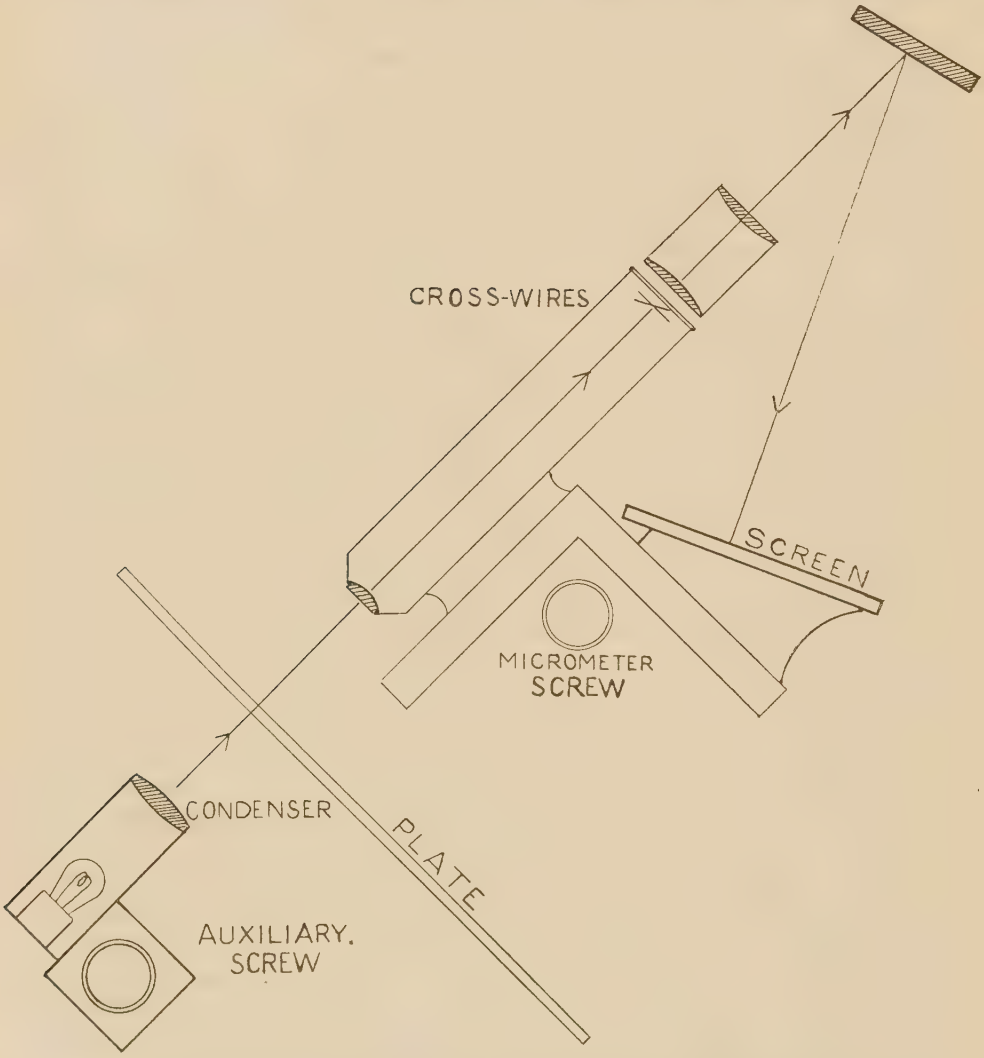
microscope objective to 3 mm. this lack is cured so far as the appearance of the image is concerned, but we still have the geometrical errors that may arise from it. A partial cure would be to use plate glass which is much more nearly plane to begin with, but for the same thickness is equally liable to bending by the contraction of the gelatine; this would, moreover, be an expensive expedient here, where 4000 plates are obtained and measured each year. Another device, which is in use at the Kapteyn Astronomical Laboratory at Groningen, is to press the photograph against a thick sheet of plate glass fixed in the measuring engine.* Dr. Bennett has suggested still another and very simple plan which we find works well. Near the centre of the plate a metal disc, about 2 cm. in diameter and coated with soft cloth, is pressed against the under side of the plate by means of a long slender lever whose other end is operated by a fine-thread screw conveniently placed. After putting the plate in the engine the observer raises the disc by whatever amount may be necessary to bring the centre into focus, the microscope having been left in the position that corresponds to the focus at the edges. This device is not always applicable since the disc and the lever may make it impossible to measure all the objects that one would need or wish to include. But this objection does not apply to parallax plates nor to certain others that we have had to deal with. It can be shown that bending the plates in this way introduces errors that are of a lower order than those that are inevitably present from other causes.

The modifications that we have added for the projection method are shown in the figure by heavy lines, the light lines indicating parts of the engine as used in the ordinary way. A six-volt automobile lamp is mounted below the plate on a nut that is operated by an auxiliary screw, of the same length and pitch as the micrometer screw but having a left-hand thread. Two equal spur gears connect these two screws, so that when the lamp has once been placed in the axis of the microscope it remains there in all positions of the latter. A condensing lens projects a beam through the plate so as to form an image of the filament upon the microscope objective. We thus obtain in the plane of the cross-wires an image of any object upon the plate surrounded by a bright background. Cross-wires and image are projected together through a positive eyepiece; they are reflected downward from a mirror a few centimetres above the eyepiece and fall upon a white matt screen as shown. For this screen we use a sheet of magnesium carbonate, which is ideal for the purpose and can be renewed whenever need be at very small expense. The screen moves with the microscope. The eyepiece and the mirror move together, but must be slid by hand over the microscope; there is a mark on the edge of the screen through which the image of the micrometer wire must approximately pass when the bisections are made.

To vary the size of the images on the screen we need not touch the microscope; we change the height of the mirror and then re-focus the eyepiece. Experience has taught us to make the images from twenty-five to thirty times their original size on the plate. If then the eye were held at

* *Publications of the Laboratory*, No. 44, 4, 1929.

the standard distance from the screen (about 10 inches, or 25 cm.) the magnification would be twenty-five or thirty, which is unduly high. But it is less fatiguing to hold the eye at a considerably greater distance than 10 inches. The average for most people with normal eyesight or for those who wear



glasses is not far from 15 inches, and for them our magnification would be about 18. Both of us like to make this distance even greater and the net magnification for us is correspondingly less.

Under these circumstances the images as we see them on the screen show about as much contrast and about as much detail as though viewed directly in the measuring microscope. To preserve the contrast no other light must be allowed to fall upon the screen, and we have accordingly enclosed the engine and the observer in a well-ventilated cabinet from which extraneous

light is excluded. Two somewhat faint lights are concentrated upon the micrometer head and the recording blank respectively, while the bright light below the plate is carefully enclosed except for the beam that goes through the condensing lens. This lower light also radiates some heat; to prevent possible thermal distortion of the guiding ways we have placed a thick sheet of plate glass above the light to lead the convection currents away from the engine.

The micrometer index is as far from the eye as is the screen, so that the observer's eyes do not have to change focus in glancing from one to the other.

There is no question as to the superiority of this mode of measuring with regard to restfulness; it seems to entail no more eye-strain or general fatigue than computing for the same length of time with a set of well-printed tables. With the ordinary method of measuring it is a rule at this observatory that no individual is to measure more than three hours a day, and this period is broken into two halves that are separated by a rest of several hours. We are for the present adhering to the same rule for the projection method, though most of the necessity for doing so has disappeared; with the projection method an observer could, without undue fatigue, measure five or six hours a day.

To subject the method to a thorough trial, Dr. Bennett has measured each of 172 parallax plates twice, first by the ordinary method and then by projection. These refer to nine parallax regions selected at random from a larger number that he had measured in the winter of 1931–32. The projection measurements were made between February 22 and April 23 of this year. In the meantime no changes had been made in the measuring engine except those that were necessary for the projection, so that whatever differences the two sets of results exhibit may safely be ascribed to the change to the projection method alone. No plate has been rejected or remeasured on account of discordance. The same weight was invariably assigned in both solutions to any one plate.

The accompanying table largely explains itself. The unit throughout is a second of arc. It will be seen that the sums of the squares of the residuals average nearly the same for the two methods, and as a consequence the two average probable errors of a parallax are likewise nearly the same. The average difference between each pair of parallaxes taken without regard to sign is $\pm$ ″·0057.

The data in the table enable us to separate plate errors from those due to inaccuracies in measurement. If we let p represent the plate error (in a sense analagous to probable error and corresponding to unit weight), and e_1 and e_2 the probable errors of measurement for the ordinary method and for the projection respectively, then from the sums of the squares of the residuals we have:

$$p^2+e_1^2=0{\cdot}455[pv_1v_1]/(\text{No. of plates } \textit{minus}\ 3),$$

$$p^2+e_2^2=0{\cdot}455[pv_2v_2]/(\text{No. of plates } \textit{minus}\ 3).$$

There are three unknowns in each solution. The quantity 0·455 is the

Star	No. of Plates	Ordinary Method		Projection Method		$[p\Delta\Delta]$
		Parallax and Prob. Error	$[pvv]$	Parallax and Prob. Error	$[pvv]$	
		″ ″		″ ″		
BS 1862	18	·000 ± ·0061	·0140	+·005 ± ·0060	·0138	·0064
BS 5787	20	+·040 ± ·0088	·0431	+·036 ± ·0088	·0427	·0093
BS 7387	16	+·004 ± ·0084	·0220	−·006 ± ·0070	·0157	·0065
BS 1451	20	+·002 ± ·0054	·0143	+·001 ± ·0058	·0169	·0067
BS 4635	20	+·018 ± ·0071	·0165	+·019 ± ·0083	·0229	·0107
PGC 516	22	+·024 ± ·0062	·0190	+·020 ± ·0049	·0133	·0064
BS 1100	19	+·014 ± ·0076	·0244	+·007 ± ·0061	·0154	·0197
BS 2503	18	−·007 ± ·0071	·0168	−·004 ± ·0057	·0108	·0097
BS 3912	19	−·007 ± ·0064	·0172	+·009 ± ·0073	·0224	·0116
Average	19	+·0098 ± ·0070	·0208	+·0097 ± ·0067	·0193	·0097

square of 0·6745. If in addition we subtract each residual in the second solution from the corresponding residual in the first solution, we get quantities Δ that are independent of sources of error inherent in the plates. The sums of $p\Delta\Delta$ are shown in the last column of the table ; from them we get :

$$e_1^2 + e_2^2 = 0{\cdot}455[p\Delta\Delta]/(\text{No. of plates } \textit{minus } 3).$$

These three expressions enable us to determine p, e_1, e_2 in each of our nine regions. Their average, or rather the quantities derived from the averages of columns 4, 6 and 7 in the table, are as follows :—

$p = \pm$ ″·0209 or ·00111 mm.,
$e_1 = \pm$ ″·0124 or ·00066 mm. (ordinary),
$e_2 = \pm$ ″·0109 or ·00058 mm. (projection).

As there are 172 observations involving 27 unknown quantities, the probable error of p, e_1 or e_2 is about 4 per cent. of these quantities. The projection method appears to give somewhat more accurate results, but the gain is only about twice what their probable errors would imply for the probable error of their difference, and we are not inclined to lay much emphasis on this apparent improvement. We may, however, safely conclude that we have not had to pay for the convenience of the projection method by any sacrifice in accuracy.

Other members of the observatory staff have tried this method, though no one else has made so formal a test as has just been described. The result of this general experience is to confirm the favourable indications that the above measurements yield.

It has been found that the projection method requires a little less average time for the measurement of a plate ; probably this gain is to be ascribed to the improvement in the matter of fatigue.

We have examined a few stellar spectrograms in this way. Without having made more than trial settings upon the lines we believe that the method may offer as great advantages here as for stellar images. Two or more observers sometimes wish to examine simultaneously and to discuss minute details in stellar spectra, comets, nebulæ and similar objects; for such purposes projection would seem to be the ideal arrangement.

Before leaving the data that these tests have afforded, it may be well to call attention to one incidental feature. We have found that the plate error is nearly twice that due to measuring, the former being on the average 0″·021, the latter about 0″·012 by either method. This is in line with much other evidence that has been accumulated in recent years by many observers, and is an instructive indication as to where improvements in photographic astrometry are to be sought. For example, a method that would halve the probable error of measurement would, after all, not be much of an improvement in the final accuracy. Squaring and adding the two errors we have just quoted we obtain 0″·0242 for the combined probable error. If we could halve the error of measurement we should get 0″·0218 for the resultant error. This is a gain of 10 per cent., and is no greater than would be effected by getting and measuring 20 per cent. more plates. If, however, some method could be devised for halving the plate error the story would be a different one; in that case the resultant error would be reduced to 0″·0159, a gain equivalent to measuring more than double the number of plates. Attention should for a time at least be focussed upon the plates rather than upon the accuracy with which they can be measured.

In the early history of photographic astrometry much emphasis was laid upon distortions of the film and upon devices for abating them. It developed, however, that this emphasis was largely misplaced, and that at that time several other sources of error, including those of measurement, were really more important. In recent years the steady improvement in methods and knowledge has again reversed the situation, until now distortion of the film may actually be a formidable source of error, possibly the most important with which we have to contend.

It is a pleasure to acknowledge our indebtedness to the National Research Council for a grant that enabled us to carry out this experiment; and to Mr. A. Klocker, mechanician at the Yale Observatory, who constructed with much skill both the measuring engine and the projection device.

Yale University Observatory:
1932 *August* 1.

THE EQUILIBRIUM OF DISTORTED POLYTROPES

(I). The Rotational Problem.

S. Chandrasekhar.

(Communicated by E. A. Milne)

§ 1. Emden's well-known researches on the equilibrium of polytropic gas spheres has been of fundamental importance in its repercussions on the modern theories of stellar structure. But it is a matter of some surprise that scarcely any serious attempt has been made to extend Emden's researches to the case of rotating gas spheres which in their non-rotating states have polytropic distributions described by the so-called Emden functions.

The problem is exceedingly simple in its classical severity and can be formulated as follows :—

We have a gas sphere in gravitational equilibrium. It is given that the total pressure P is related to the density ρ by means of the relation

$$P = K\rho^{1+\frac{1}{n}}. \tag{1}$$

This defines for the non-rotating gas sphere a distribution of density and pressure governed by the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{2}*$$

—*Emden's differential equation of index n.* Now the gas sphere is set rotating at a constant small angular velocity ω. *The problem is to determine the shape and density distribution in such a gas sphere and explicitly to relate the structure of such rotating masses with the Emden functions which describe the polytropic non-rotating state.*

§ 2. In treating this problem we shall assume that the rotation is so slow that the configurations are only slightly oblate. In other words, the purpose of this paper is to specify *completely* those configurations for the polytropic model which correspond to the Maclaurin spheroids † in the case of the "incompressible rotating stellar masses."

It is thus clear that the point of view and the problem here is different from that considered by Jeans under the heading "Adiabatic-model" in his book on *The Problems of Cosmogony and Stellar-dynamics* (p. 165). The problem treated by Jeans is to enumerate the complete sequence of the geometry of the configurations for the whole range of ω. Further, the analysis of Jeans is to establish a general result that a gas sphere rotating as a rigid body can break up in two distinct ways—either by fission into two detached masses or by a process of equatorial break-up after assuming

* The significance of ξ and θ are explained later. † For small ω.

a lenticular shape as in the Roche model—according as the central condensation is "weak" or "pronounced." We shall not enter into such questions—which of course are of fundamental significance in any problem of cosmogony—but confine ourselves to the comparatively more simple and elementary problem of specifying completely the "polytropic Maclaurin spheroids." Also the fundamental mathematical point of view will be different from that of Jeans. We will explicitly relate the geometry and the physical properties of these configurations with the Emden functions describing the polytropic non-rotating gas spheres.

Substantially the case $n=3$ has been treated already by Milne * and von Zeipel.† Indeed it was in this connection that von Zeipel discovered his fundamental theorem.‡ The method of solution adopted in the sequel will in main be that of Milne, to whose memoir the present paper owes a great deal.

§ 3. *Equations of the Problem.*—The equations of mechanical equilibrium are, taking the Z-axis as the axis of rotation,

$$\left.\begin{aligned} \frac{\partial P}{\partial x} &= \rho\frac{\partial V}{\partial x} + \rho\omega^2 x, \\ \frac{\partial P}{\partial y} &= \rho\frac{\partial V}{\partial y} + \rho\omega^2 y, \\ \frac{\partial P}{\partial z} &= \rho\frac{\partial V}{\partial z}, \end{aligned}\right\} \tag{3}$$

where V is the gravitational potential, which of course satisfies Poisson's equation

$$\sum_{x,\,y,\,z}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = -4\pi G\rho. \tag{4}$$

Introducing polar co-ordinates r, θ, ϕ and neglecting ϕ on account of symmetry we find that

$$\left.\begin{aligned} \frac{\partial P}{\partial r} &= \rho\frac{\partial V}{\partial r} + \rho\omega^2 r(1-\mu^2), \\ \frac{\partial P}{\partial \mu} &= \rho\frac{\partial V}{\partial \mu} - \rho\omega^2 r^2\mu, \end{aligned}\right\} \tag{3'}$$

where $\mu = \cos\theta$. Also (4) in these variables reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left((1-\mu^2)\cdot\frac{\partial V}{\partial \mu}\right) = -4\pi G\rho. \tag{4'}$$

From (3′) and (4′) we deduce the fundamental equation of the problem:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left(\frac{1-\mu^2}{\rho}\frac{\partial P}{\partial \mu}\right) = -4\pi G\rho + 2\omega^2, \tag{5}$$

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* E. A. Milne, *M.N.*, 83, 118, 1923. † H. von Zeipel, *M.N.*, 84, 665, 684, 1924.

‡ For a general discussion of the whole problem we refer the reader to Milne's article in the *Handbuch der Astrophysik*, Band III/1, p. 235.

where

$$P = K\rho^{1+\frac{1}{n}}. \tag{6}$$

§ 4. If we introduce the new variables

$$\rho = \lambda\Theta^n; \qquad P = \lambda^{1+\frac{1}{n}} . K . \Theta^{n+1}, \tag{7}$$

(5) reduces to

$$\frac{\lambda^{\frac{1}{n}-1}K(n+1)}{4\pi G}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)\right\} = -\Theta^n+\frac{\omega^2}{2\pi G\lambda}. \tag{8}$$

Put

$$r = \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}}\xi \tag{9}$$

and

$$v = \frac{\omega^2}{2\pi G\lambda}. \tag{10}$$

We get

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right) = -\Theta^n+v. \tag{11}$$

§ 5. *Non-rotating Configuration.*—If the gas sphere is non-rotating then ω and therefore v is zero. If, further, now

$$\rho = \lambda\theta^n, \tag{7'}$$

we see that θ satisfies the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{12}$$

—Emden's differential equation of index *n*.

§ 6. *Solution for the Rotating Gas Sphere.*—We will now seek a solution of (11) in terms of those of (12), and indeed we will assume the following form for our solution:—

$$\Theta = \theta + v\Psi + v^2\Phi + \ldots \tag{13}$$

We shall work consistently only up to the first order in v, i.e. we consider only such slow rotations that the effects arising from ω^4 can be neglected. Ψ then should satisfy the differential equation, remembering that θ is a spherically symmetrical function and therefore independent of μ,

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Psi}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right) = -n\theta^{n-1}\Psi+1. \tag{14}$$

Now we shall assume for Ψ the following form: *

$$\Psi = \psi_0(\xi)+\sum_{j=1}^{\infty}A_j\psi_j(\xi)P_j(\mu), \tag{15}$$

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* For a formal justification of this assumption see von Zeipel (*loc. cit.*), p. 691.

where the $P_j(\mu)$'s are the Legendre functions of the various indices, the function with index j satisfying the differential equation

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial P_j}{\partial\mu}\right)+j(j+1)P_j=0. \tag{16}$$

Further, the ψ_j's are functions of ξ only.

Substituting (15) in (14), and using (16) and equating coefficients of P_j, we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)=-n\theta^{n-1}\psi_0+1, \tag{17}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)=\left(\frac{j(j+1)}{\xi^2}-n\theta^{n-1}\right)\psi_j, \tag{18}$$

$$(j=1, 2, \ldots).$$

So far the A_j's are arbitrary. To determine them we must evaluate the potential V. For equation (5) (from which (14) is deduced) contains no explicit reference to the potential and is the same whatever the external gravitational field. To remove this indeterminateness we must use the solution found (with the arbitrary A_j's) to calculate the potential arising from the matter and then determine the A_j's such that the equations of equilibrium (3′) are satisfied.

Poisson's equation in the ξ, μ variables takes the form

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial V}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right)$$
$$=-(n+1)K\lambda^{\frac{1}{n}}\left[\theta^n+n\theta^{n-1}v\left\{\psi_0+\sum_j A_j\psi_j P_j\right\}\right]. \tag{19}$$

We develop V in the form (to the first order in v)

$$V=U+v\left\{V_0(\xi)+\sum_j V_j(\xi)P_j(\mu)\right\}, \tag{20}$$

where U is the potential of the non-rotating configuration. Substitution of (20) in (19) yields on equating the coefficients of $P_j(\mu)$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dU}{d\xi}\right)=-R\theta^n, \tag{21}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_0}{d\xi}\right)=-Rn\theta^{n-1}\psi_0, \tag{22}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right)=\frac{j(j+1)}{\xi^2}V_j-Rn\theta^{n-1}A_j\psi_j, \tag{23}$$

where we have used the abbreviation

$$R=(n+1)K\lambda^{\frac{1}{n}}.$$

Remembering that θ satisfies the Emden equation with index n we deduce from (21) that

$$U=R\theta+\text{constant}. \tag{24}$$

Using the differential equation (17) defining ψ_0, we derive from (22) that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_0}{d\xi}\right)=R\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)-1\right]$$
$$=R\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}(\psi_0-\tfrac{1}{6}\xi^2)\right)\right].$$

Hence

$$V_0=R(\psi_0-\tfrac{1}{6}\xi^2)+\text{constant}. \tag{25}$$

Similarly from (23) and (18) we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right)-\frac{j(j+1)}{\xi^2}V_j=A_jR\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)-\frac{j(j+1)}{\xi^2}\psi_j\right]. \tag{26}$$

A particular solution of (26) is

$$V_j=RA_j\psi_j+\text{constant}. \tag{27}$$

The general solution is obtained by adding any *regular* solution of the equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right)-\frac{j(j+1)}{\xi^2}V_j=0, \tag{28}$$

which is $RB_j\xi^j$ where B_j is arbitrary. Hence

$$V_j=R(A_j\psi_j+B_j\xi^j)+\text{constant}. \tag{29}$$

Combining the results (24), (25) and (29) we get after some minor rearrangement of the terms (to the first order in v)

$$V=R\left[\Theta+v\left\{\sum_{j=1}^{\infty}B_j\xi^jP_j(\mu)-\tfrac{1}{6}\xi^2\right\}\right]. \tag{30}$$

We have to substitute now (30) in (3′), which in the ξ, μ variables takes the form

$$\frac{dP}{d\xi}=\rho\frac{\partial V}{\partial\xi}+\frac{2}{3}\rho\omega^2\xi\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right](1-P_2(\mu)).$$

Substituting (30) in the above equation and remembering that

$$P=\lambda^{1+\frac{1}{n}}\,.\,K\,.\,\Theta^{n+1},$$

and equating coefficients of $P_j(\mu)$ we find that

$$B_j=0, \qquad j\neq 2$$

and

$$B_2=\tfrac{1}{6}. \tag{31}$$

Finally, we obtain

$$V=R(\Theta-\tfrac{1}{6}v(\xi^2-P_2(\mu)\xi^2))+\text{constant}. \tag{32}$$

We have still to ensure that V is the *actual* potential arising from the mass. This will determine the A_j's.

A little consideration shows that (compare Milne (*loc. cit.*), p. 134) to the

order of accuracy we are working, the potential (and its derivative) given by (32) should be continuous with an expression of the type

$$V_{\text{external}} = R\left[\frac{C_0}{\xi} + v\sum_{j=1}^{\infty}\frac{C_j}{\xi^{j+1}}P_j(\mu)\right] + \text{constant} \tag{33}$$

on a sphere of radius ξ_1 the first zero of the Emden's function with index n. Comparing the "inner" and the "external" potentials at $\xi = \xi_1$ and also their derivatives we obtain

$$A_j = C_j = 0, \qquad j \neq 2.$$

But, if $j = 2$, we get

$$\left.\begin{aligned}\frac{C_2}{{\xi_1}^3} &= A_2\psi_2(\xi_1) + \tfrac{1}{6}{\xi_1}^2,\\ -\frac{3C_2}{{\xi_1}^4} &= A_2\psi_2'(\xi_1) + \tfrac{1}{3}\xi_1.\end{aligned}\right\} \tag{34}$$

This gives

$$A_2 = -\frac{5}{6}\frac{{\xi_1}^2}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}. \tag{35}$$

Hence the solution to the problem is given by

$$\Theta = \theta + v\left[\psi_0(\xi) - \frac{5}{6}\frac{{\xi_1}^2}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}\psi_2(\xi)P_2(\mu)\right], \tag{36}$$

where ψ_0 and ψ_2 satisfy the differential equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1, \tag{37$_1$}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\psi_2, \tag{37$_2$}$$

which are for purposes of numerical integration more conveniently written as

$$\frac{d^2\eta_0}{d\xi^2} = -n\theta^{n-1}\eta_0 + \xi\,; \qquad \frac{d^2\eta_2}{d\xi^2} = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\eta_2, \tag{37$'$}$$

where

$$\eta_{0,2} = \xi\psi_{0,2}. \tag{37$''$}$$

§ 7. *Expansion, Ellipticity and Oblateness.*—The boundary ξ_0 is given by $\Theta = 0$, and hence by (36)

$$\xi_0 = \xi_1 + \frac{v}{|\,\theta_1'\,|}\left[\psi_0(\xi_1) - \frac{5}{6}\frac{{\xi_1}^2\psi_2(\xi_1)P_2(\mu)}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}\right]. \tag{38}$$

Thus there is an expansion of the star as a whole of amount $v\psi_0(\xi_1)/|\,\theta_1'\,|$ and superposed on this an ellipticity. At the equator $P_2(\mu) = -\frac{1}{2}$ and at the poles $P_2(\mu) = +1$. Hence we get for the *oblateness* of the boundary the general expression

$$\sigma = \frac{5}{4}\frac{v}{|\,\theta_1'\,|}\cdot\frac{\xi_1\psi_2(\xi_1)}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}. \tag{39}$$

§ 8. *Mass Relation.*—The mass is given by

$$M = 2\pi \iint \rho r^2 dr d\mu.$$

The ellipticity term does not clearly contribute to the mass on the average. Introducing the ξ, θ variables in the above integral we have

$$M = 4\pi \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2} \lambda \int_0^{\xi_1+d\xi_1} (\theta^n + n\theta^{n-1} v\psi_0)\xi^2 d\xi,$$

which to the first order in v can be replaced by

$$M = 4\pi \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-\frac{1}{3}}\right]^{3/2} \left\{\int_0^{\xi_1} \theta^n \xi^2 d\xi + v\int_0^{\xi_1} n\theta^{n-1}\psi_0 \xi^2 d\xi\right\}.$$

Now

$$\int_0^{\xi_1} \theta^n \xi^2 d\xi = -\int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_1,$$

$$\int_0^{\xi_1} n\theta^{n-1}\psi_0 \xi^2 d\xi = -\int_0^{\xi_1} \left\{\frac{d}{d\xi}\left(\xi^2 \frac{d\psi_0}{d\xi}\right) - \xi^2\right\} d\xi$$

$$= -\xi_1^2 \psi_0'(\xi_1) + \tfrac{1}{3}\xi_1^3.$$

Hence

$$M = -4\pi \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{3-n}{3n}}\right]^{3/2} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_1 \cdot \left[1 + v\frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\theta_1'|}\right]. \tag{40}$$

If $v = 0$, we have for the case of non-rotating stars

$$M_0 = -4\pi \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{3-n}{3n}}\right]^{3/2} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_1. \tag{41}$$

Hence the "mass relation" for two gas spheres with *equal central densities*—one rotating with an angular velocity ω and the other non-rotating—is

$$M_\omega = M_0 \cdot \left[1 + v\frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\theta_1'|}\right]. \tag{42}$$

Hence the rotating configuration has a *greater mass*, as indeed we should expect on general grounds.*

§ 9. *Volume and Relation between Mean and Central Density.*—The volume N of the configuration is clearly given by

$$N = \frac{4\pi}{3}\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2} \xi_1^3 + \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2} \cdot \int_0^{2\pi}\int_{-1}^{+1} (\xi_0 - \xi_1)\xi_1^2 d\mu d\phi,$$

which by (38) yields

$$N = \frac{4}{3}\pi \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{3/2} \xi_1^3 \cdot \left[1 + v \cdot \frac{3\psi_0(\xi_1)}{\xi_1 |\theta_1'|}\right]. \tag{43}$$

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* Compare Milne's article in the *Handbuch* ("General Effects of Rotation," p. 236).

Using the mass relation (40) we get for the mean density ρ_m the formula

$$\rho_m = -3\lambda \frac{1}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 \cdot \frac{1 + v \cdot \dfrac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\theta_1'|}}{1 + v \cdot \dfrac{3\psi_0(\xi_1)}{\xi_1 |\theta_1'|}},$$

or to the order of accuracy we are working we have, remembering that $\lambda = \rho_c$ (θ and Θ are both unity at the centre),

$$\rho_m = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 \rho_c \cdot \left[1 + v\frac{\frac{1}{3}\xi_1^2 - \psi_0'(\xi_1)\xi_1 - 3\psi_0(\xi_1)}{\xi_1 |\theta_1'|}\right]. \tag{44}$$

If $v = 0$, we get the well-known formula for the non-rotating polytropes

$$\rho_m = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 \rho_c. \tag{45}$$

From (44) we deduce (to the first order) that

$$v = \frac{\omega^2}{2\pi G\rho_c} = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 \cdot \frac{\omega^2}{2\pi G\rho_m}. \tag{46}$$

Formulæ (38, (39), (40), (42), (43) can all now be expressed in terms of ρ_m (instead of ρ_c) by means of (46).

§ 10. *Numerical Integration of the Differential Equations.*—Thus we see that the structure of a slowly rotating polytropic gas configuration is completely specified when the pair of differential equations ($37_{1,\,2}$) are solved. There is just one case where integration can be effected at once. The equation for ψ_0 for the index $n = 1$ can be written as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d(\psi_0 - 1)}{d\xi}\right) = -(\psi_0 - 1), \tag{47}$$

which is just Emden's equation with index 1. Remembering the boundary conditions that ψ_0 and ψ_0' are to be zero at the origin, we see that

$$\psi_0 = 1 - \frac{\sin\xi}{\xi}, \tag{48}$$

is the required solution. [By (48) we see that

$$\psi_0(\xi_1) = 1\,; \qquad \psi_0'(\xi_1) = \frac{1}{\pi}\,; \qquad \xi_1 = \pi. \tag{48'}$$

The mass relation (42) and the volume relation (43) therefore take for this case the neat forms

$$M_\omega = M_0[1 + v(\tfrac{1}{3}\pi^2 - 1)], \tag{48''}$$

$$N_\omega = N_0[1 + 3v]. \tag{48'''}$$

In all other cases numerical integration must be adopted. To do this we must have a power series in ξ for ψ_0 and ψ_2 at the origin, and with a start

thus made the integration has to be continued by any of the known standard methods.

Now it is seen that near the origin Emden's equation with index n has the expansion *

$$\theta = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{8n^2 - 5n}{3 \cdot 7!}\xi^6 + \ldots \tag{49}$$

which yields for θ^{n-1} the expansion

$$\theta^{n-1} = 1 - \frac{(n-1)}{6}\xi^2 + \frac{(n-1)(4n-5)}{180}\xi^4 - \ldots \tag{50}$$

Substituting this in equations $(37_{1,\,2})$, and assuming for ψ_0 and ψ_2 power series of the type

$$\psi_{0,\,2} = a\xi^2 + b\xi^3 + c\xi^4 + \ldots, \tag{51}$$

and solving, we obtain the following expansions for ψ_0 and ψ_2 near the origin :—

$$\psi_0 = \frac{1}{6}\xi^2 - \frac{n}{120}\xi^4 + \frac{n(13n-10)}{42 \cdot 360}\xi^6 - \frac{n(90n^2 - 157n + 70)}{72 \cdot 42 \cdot 360}\xi^8 + \ldots \tag{52}$$

$$\psi_2 = \xi^2 - \frac{n}{14}\xi^4 + \frac{n(10n-7)}{42 \cdot 36}\xi^6 - \frac{n(308n^2 - 503n + 210)}{42 \cdot 36 \cdot 330}\xi^8 + \ldots \tag{53}$$

The numerical integration was carried out for the cases $n = 1,\ 1{\cdot}5,\ 2,\ 3,\ 4$, and the functions ψ_0 and ψ_2 are tabulated in Tables I–V appended to the end of this paper. At the bottom of each table the value of ψ_0 and ψ_2 and their derivatives at ξ_1—the first zero of Emden's equation—are given. The method of integration adopted is the one attributed to Adams and sketched at the end of the second of von Zeipel's papers referred to at the outset. In the following Table VI the values of $\psi_0(\xi_1)$, $\psi_0'(\xi_1)$, $\psi_2(\xi_1)$, $\psi_2'(\xi_1)$, ξ_1 and θ_1' are given.

TABLE VI

n	1	1·5	2	3	4
ξ_1	3·14159	3·6538	4·3529	6·8968	14·9715
$-\theta_1'$	0·31831	0·20330	0·12725	0·04243	0·0080181
$\psi_0(\xi_1)$	1·00000	1·2942	1·9153	5·8380	33·5327
$\psi_0'(\xi_1)$	0·31831	0·6364	0·9961	2·0391	4·8812
$\psi_2(\xi_1)$	4·55940	4·7820	5·6431	11·2780	46·5444
$\psi_2'(\xi_1)$	0·42060	1·1495	1·7559	3·0409	6·1766

§ 11. With the use of the values given in the above table we can now numerically evaluate many of the "coefficients" occurring in the formulæ

* See the introduction (by D. H. Sadler) to the *Mathematical Tables*, vol. ii, on "Emden Functions," issued by the British Association for the Advancement of Science.

of §§ 7, 8 and 9. We shall now give the precise forms for the cases where the numerical integration has been effected.

Firstly, equation (36), expressing Θ, takes the following forms :—

$$\left.\begin{aligned}
\Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}5483\psi_2(\xi)P_2(\mu)], \quad n = 1\\
\Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}5999\psi_2(\xi)P_2(\mu)], \quad n = 1{\cdot}5\\
\Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}6426\psi_2(\xi)P_2(\mu)], \quad n = 2\\
\Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}72325\psi_2(\xi)P_2(\mu)], \quad n = 3\\
\Theta &= \theta + v[\psi_0(\xi) - 0{\cdot}8048\psi_2(\xi)P_2(\mu)], \quad n = 4
\end{aligned}\right\}. \qquad (54)$$

The equations of the boundary ξ_0 are :

$$\left.\begin{aligned}
\xi_0 &= 3{\cdot}1416 + 3{\cdot}1416v[\ 1 - 2{\cdot}5000P_2(\mu)], \quad n = 1\\
\xi_0 &= 3{\cdot}6538 + 4{\cdot}9188v[\ 1{\cdot}2942 - 2{\cdot}8687P_2(\mu)], \quad n = 1{\cdot}5\\
\xi_0 &= 4{\cdot}3529 + 7{\cdot}8585v[\ 1{\cdot}9153 - 3{\cdot}6260P_2(\mu)], \quad n = 2\\
\xi_0 &= 6{\cdot}8968 + 23{\cdot}568v\ [\ 5{\cdot}8380 - 8{\cdot}1568P_2(\mu)], \quad n = 3\\
\xi_0 &= 14{\cdot}9715 + 124{\cdot}718v\ [33{\cdot}533 - 37{\cdot}460P_2(\mu)], \quad n = 4
\end{aligned}\right\}. \qquad (55)$$

Remembering that $P_2(\mu) = -\frac{1}{2}$ at the equator and $+1$ at the poles we obtain from (55) the following table giving *the fractional elongation at the equator, the fractional contraction at the poles and the oblateness* as expressed by equation (39):—

TABLE VII

Geometry of the Boundary

n	Fractional Elongation at the Equator	Fractional Contraction at the Poles	Oblateness σ
1	$2{\cdot}2500v$	$1{\cdot}5000v$	$3{\cdot}7500v$
1·5	$3{\cdot}6730v$	$2{\cdot}1195v$	$5{\cdot}7926v$
2	$6{\cdot}7309v$	$3{\cdot}0884v$	$9{\cdot}8194v$
3	$33{\cdot}888v$	$7{\cdot}9240v$	$41{\cdot}811v$
4	$435{\cdot}35v$	$32{\cdot}714v$	$468{\cdot}07v$

The mass relations are :

$$\left.\begin{aligned}
M_\omega &= M_0 \,.\, (1 + 2{\cdot}290v), \quad n = 1\\
M_\omega &= M_0 \,.\, (1 + 2{\cdot}860v), \quad n = 1{\cdot}5\\
M_\omega &= M_0 \,.\, (1 + 3{\cdot}575v), \quad n = 2\\
M_\omega &= M_0 \,.\, (1 + 6{\cdot}123v), \quad n = 3\\
M_\omega &= M_0 \,.\, (1 + 13{\cdot}632v), \quad n = 4
\end{aligned}\right\}. \qquad (56)$$

The volume relations are :

$$\left.\begin{aligned}
N_\omega &= N_0 \,.\, (1 + 3v), \quad n = 1\\
N_\omega &= N_0 \,.\, (1 + 5{\cdot}227v), \quad n = 1{\cdot}5\\
N_\omega &= N_0 \,.\, (1 + 10{\cdot}373v), \quad n = 2\\
N_\omega &= N_0 \,.\, (1 + 59{\cdot}849v), \quad n = 3\\
N_\omega &= N_0 \,.\, (1 + 838{\cdot}00v), \quad n = 4
\end{aligned}\right\}. \qquad (57)$$

The relations between the mean and the central densities are :

$$\left.\begin{aligned}
\rho_c &= \rho_m \times \ 3{\cdot}2899[1 + \ 0{\cdot}710v], \quad n = 1 \\
\rho_c &= \rho_m \times \ 5{\cdot}9907[1 + \ 2{\cdot}367v], \quad n = 1{\cdot}5 \\
\rho_c &= \rho_m \times 11{\cdot}4025[1 + \ 6{\cdot}797v], \quad n = 2 \\
\rho_c &= \rho_m \times 54{\cdot}1825[1 + 53{\cdot}724v], \quad n = 3 \\
\rho_c &= \rho_m \times 622{\cdot}408 \ [1 + 814{\cdot}37v], \quad n = 4
\end{aligned}\right\}. \tag{58}$$

§ 12. *Comparison of Configurations with equal Mass.*—The above sections (in particular § 11) give a complete answer to the problem formulated at the outset in § 1. But it is of interest to compare two configurations, one "stationary" and the other rotating with a slow angular velocity ω, both having the *same mass*. Now for this problem to have a meaning, the radius of the non-rotating polytrope must be determined when the mass is given. As is well known (and as is clear from equation (41)) this is not the case when $n = 3$. Hence we should expect (and indeed as it turns out subsequently) that $n = 3$ must be of the nature of a singularity with respect to this problem.

We have already seen that if a rotating and a non-rotating configuration are to have the same central density, then the non-rotating configuration has a smaller mass. To secure the same mass we must alter the central density of one of them—say the rotating one. The problem then is to find the fractional increase (or decrease) in λ (*i.e.* the central density) such that the masses of the two configurations are the same, *i.e.* we have to find $\delta\lambda$ such that

$$M(\lambda + \delta\lambda, \omega) = M(\lambda, 0). \tag{59}$$

From (41) we easily find by differentiation with respect to λ that

$$\frac{\delta\lambda}{\lambda} = -\frac{2n}{\dfrac{(3-n)}{v}\dfrac{|\theta_1'|}{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)} - 3(n-1)}. \tag{60}$$

If $n \neq 3$ we can rewrite the above relation approximately to the first order as

$$\frac{\delta\lambda}{\lambda} = -v \cdot \frac{2n}{3-n}\frac{\frac{1}{3}\xi_1 - \psi_0'(\xi_1)}{|\theta_1'|}. \tag{61}$$

It is at once clear from the above formula that *if* $n > 3$ *the central density of the rotating configuration is greater than that of the non-rotating configuration, while if* $n < 3$ *the converse is true.*

Of course the above result is precisely what we should expect. For if the central densities of the two configurations are equal the non-rotating configuration has a *smaller* mass, and as we have

$$M_0 \propto \lambda^{\frac{3-n}{2n}}, \tag{62}$$

to *increase* the mass of the non-rotating sphere we should increase or decrease λ according as n is less than or greater than 3. This is just what (61) shows. When $n = 3$ the radius of the non-rotating configuration is indeterminate

and the question "the fractional change in the central density" has no meaning, for the radius of the polytrope $n=3$ is first determined only when the mean density (and therefore also the central density) is given (in addition to M).

If $n \neq 3$ the change in the central density is proportional to v and hence *formulæ* (54), (55), (57), (58) *and the results of* Table VI *continue to be true (to the first order in* v*) when we are comparing two configurations with equal mass.*

Finally, the following short table gives precise values for $\frac{\delta\lambda}{\lambda}$:—

n	1	1·5	2	3	4
$-\delta\lambda/\lambda$	$2{\cdot}290v$	$5{\cdot}720v$	$14{\cdot}299v$	...	$-109{\cdot}06v$

§ 13. *General Discussion.*—There are a number of points which can now be discussed in connection with the above calculations and their bearing on actual physical facts, but I shall make only some brief comments and reserve more detailed references to possible applications for a separate communication after treating the closely related problem of the double stars.

(1) Though the problem has been formulated in an abstract form and the whole calculation proceeds on the assumption that there exists a relation $P=K\rho^{1+\frac{1}{n}}$ between the total pressure and density, it can actually be shown that the whole investigation applies to the more general "standard model."

(2) A glance at Table VII shows that as the central condensation of the configuration increases the fractional elongation at the equator and the fractional contraction at the poles increase rapidly for a given angular velocity ω. But what is noteworthy is that the fractional elongation at the equator increases much more rapidly than the fractional contraction at the poles. Thus the ratio of the fractional elongation at the equator to the fractional contraction at the poles increases from 1·5 for $n=1$ to 13·3 for $n=4$. This is precisely what one would expect if for *large values* of ω the configurations with large central condensations should tend to assume lenticular forms for equatorial break-up.

(3) In § 12 we considered the fractional change in the central density ($\delta\lambda/\lambda$) which takes place when a gas sphere is set rotating with a slow angular velocity ω. It was found that if $n<3$, $\delta\lambda/\lambda$ is *negative*, while if $n>3$, $\delta\lambda/\lambda$ is *positive*. Thus from this standpoint configurations with "large" central condensations behave quite differently from configurations with comparatively weak central condensations. One is tempted to conjecture that these two distinct different *initial* behaviours of rotating masses correspond to the two distinct types of break-up—namely, the fissional break-up and the equatorial break-up—which occur when ω gets large. But this conjecture, though fascinating, is by no means well founded.

In conclusion, I wish to record my thanks to Professor N. Bohr for

allowing me the very valuable privileges of his Institute, where the above work was carried out. My thanks are also due to Professor E. A. Milne for his interest and advice.

APPENDIX

Solutions of the differential equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1,$$

and

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\psi_2,$$

for n = 1, 1·5, 2, 3 *and* 4.

In the following tables, in addition to ψ_0 and ψ_2, $n\theta^{n-1}$ is also tabulated. The values of the Emden functions were taken from *Mathematical Tables*, vol. ii, on "Emden Functions," issued by L. J. Comrie on behalf of the British Association for the Advancement of Science. Only as many decimal figures are retained as are regarded to be reliable.

TABLE I

Index $n = 1$

ξ	$\frac{6}{\xi^2} - 1$	ψ_0	ψ_2
0·	∞	0·	0·
0·2	149·0000	0·00665	0·039886
0·4	36·5000	0·02645	0·158180
0·6	15·6667	0·05893	0·350835
0·8	8·3750	0·10330	0·611258
1·0	5·0000	0·15853	0·93053
1·2	3·16666	0·22330	1·2977
1·4	2·06122	0·28611	1·7001
1·6	1·34375	0·37527	2·1239
1·8	0·85185	0·45897	2·5544
2·0	0·50000	0·54535	2·9767
2·2	0·23967	0·63250	3·3759
2·4	0·04167	0·71856	3·7380
2·6	−0·11243	0·80173	4·0499
2·8	−0·23469	0·88036	4·3002
3·0	−0·33333	0·95296	4·4795
3·2	−0·41406	...	4·5804
3·4	−0·48097	...	4·5982

$\xi_1 = 3{\cdot}14159,$

$\psi_0(\xi_1) = 1{\cdot}00000, \qquad \psi_2(\xi_1) = 4{\cdot}55940,$

$\psi_0'(\xi_1) = 0{\cdot}31831, \qquad \psi_2'(\xi_1) = 0{\cdot}42060.$

Table II

Index $n = 1{\cdot}5$

ξ	$1{\cdot}5\theta^{\frac{1}{2}}$	ψ_0	ψ_2
0·0	1·5000	0·	0·
0·2	1·4950	0·0066467	0·0398291
0·4	1·4801	0·026351	0·157289
0·6	1·4556	0·058423	0·346477
0·8	1·4217	0·101785	0·59813
1·0	1·3790	0·155061	0·90040
1·2	1·3281	0·21669	1·23978
1·4	1·2698	0·28506	1·6022
1·6	1·2046	0·35862	1·9741
1·8	1·1332	0·43601	2·3430
2·0	1·0564	0·51612	2·6985
2·2	0·97449	0·59822	3·0324
2·4	0·88802	0·68194	3·3401
2·6	0·79699	0·76739	3·6197
2·8	0·70067	0·85511	3·8714
3·0	0·59787	0·94618	4·0992
3·2	0·48403	1·04226	4·3095
3·4	0·35286	1·14581	4·5121
3·6	0·15797	1·26074	4·7218

$\xi_1 = 3{\cdot}6538,$

$\psi_0(\xi_1) = 1{\cdot}2942, \qquad \psi_2(\xi_1) = 4{\cdot}7820,$

$\psi_0'(\xi_1) = 0{\cdot}6364, \qquad \psi_2'(\xi_1) = 1{\cdot}1495.$

Table III

Index $n = 2$

ξ	2θ	ψ_0	ψ_2
0·	2·00000	0·	0·
0·2	1·98672	0·0066401	0·039773
0·4	1·94751	0·026249	0·15641
0·6	1·88419	0·057935	0·34226
0·8	1·79959	0·100359	0·58572
1·0	1·69731	0·152110	0·87277
1·2	1·58134	0·21126	1·1886
1·4	1·45582	0·27629	1·5190
1·6	1·32472	0·34605	1·8519
1·8	1·19165	0·41957	2·1778
2·0	1·05967	0·49641	2·4904
2·2	0·93130	0·57652	2·7862
2·4	0·80842	0·66030	3·0643
2·6	0·69237	0·74845	3·3263
2·8	0·58398	0·84195	3·5751
3·0	0·48365	0·94200	3·8151
3·2	0·39145	1·0500	4·0515
3·4	0·30720	1·1675	4·2898
3·6	0·23051	1·2963	4·5361
3·8	0·16086	1·4381	4·7966
4·0	0·09768	1·5949	5·0776
4·2	0·04032	1·7691	5·3855
4·4	...	1·9629	5·7274

$\xi_1 = 4{\cdot}3529,$

$\psi_0(\xi_1) = 1{\cdot}9153, \qquad \psi_2(\xi_1) = 5{\cdot}6431,$

$\psi_0'(\xi_1) = 0{\cdot}9961, \qquad \psi_2'(\xi_1) = 1{\cdot}7559.$

TABLE IV

Index $n = 3$

ξ	$3\theta^2$	ψ_0	ψ_2
0·	3·000	0·	0·
0·2	2·960	0·0066271	0·039657
0·4	2·846	0·026050	0·15470
0·6	2·668	0·057010	0·33425
0·8	2·444	0·09768	0·56286
1·0	2·194	0·14637	0·8237
1·2	1·932	0·20142	1·1015
1·4	1·676	0·26168	1·3841
1·6	1·434	0·32655	1·6632
1·8	1·215	0·39596	1·9341
2·0	1·019	0·47021	2·195
2·2	0·8486	0·5499	2·448
2·4	0·7023	0·6360	2·693
2·6	0·5785	0·7292	2·935
2·8	0·4743	0·8306	3·175
3·0	0·3870	0·9412	3·419
3·2	0·3147	1·0618	3·668
3·4	0·2549	1·1932	3·924
3·6	0·2055	1·3364	4·192
3·8	0·1648	1·4920	4·472
4·0	0·1314	1·6607	4·767
4·2	0·1040	1·8431	5·078
4·4	0·08168	2·0398	5·407
4·6	0·06352	2·2513	5·753
4·8	0·04876	2·4780	6·121
5·0	0·03682	2·7203	6·506
5·2	0·02730	2·9787	6·916
5·4	0·01971	3·2532	7·346
5·6	0·01376	3·5443	7·798
5·8	0·00853	3·8520	8·272
6·0	0·00574	4·1765	8·768
6·2	0·00325	4·5180	9·288
6·4	0·00155	4·8760	9·829
6·6	0·00052	5·2510	10·39
6·8	0·00005	5·6429	10·98
7·0	...	6·0510	11·59

$$\xi_1 = 6{\cdot}8969,$$

$$\psi_0(\xi_1) = 5{\cdot}8380, \qquad \psi_2(\xi_1) = 11{\cdot}2780,$$

$$\psi_0'(\xi_1) = 2{\cdot}0391, \qquad \psi_2'(\xi_1) = 3{\cdot}0409.$$

TABLE V

Index $n=4$

ξ	$4\theta^3$	ψ_0	ψ_2
0·	4·0000	0·	0·
0·2	3·9209	0·0066140	0·0395434
0·4	3·6976	0·025857	0·153027
0·6	3·3651	0·056144	0·32671
0·8	2·9682	0·095460	0·54239
1·0	2·5514	0·141909	0·78215
1·2	2·1490	0·19413	1·03149
1·4	1·7824	0·25144	1·2809
1·6	1·4625	0·31378	1·5255
1·8	1·1915	0·37044	1·7636
2·0	0·96690	0·45543	1·9962
2·2	0·78328	0·53624	2·2254
2·4	0·63448	0·62487	2·4536
2·6	0·51458	0·72217	2·6837
2·8	0·41824	0·82892	2·9181
3·0	0·34076	0·94583	3·1591
3·2	0·27853	1·07355	3·4087
3·4	0·22845	1·2126	3·6683
3·6	0·18807	1·3636	3·9394
3·8	0·15533	1·5268	4·2230
4·0	0·12869	1·7027	4·5200
4·2	0·10695	1·8916	4·8311
4·4	0·08916	2·0938	5·1569
4·6	0·07454	2·3096	5·4979
4·8	0·06251	2·5390	5·8546
5·0	0·05253	2·7825	6·2270
5·2	0·04424	3·0400	6·6161
5·6	0·03159	3·5980	7·4434
6·0	0·02271	4·2140	8·3380
6·4	0·01639	4·8886	9·3008
6·8	0·01190	5·6223	10·3326
7·2	0·008648	6·4156	11·4337
7·6	0·006285	7·2685	12·6045
8·0	0·004566	8·1811	13·8451
8·4	0·003306	9·1535	15·1558
8·8	0·002384	10·1856	16·5364
9·2	0·001707	11·2773	17·9870
9·6	0·001211	12·4301	19·5076
10·0	0·000850	13·6407	21·0980
10·4	0·000588	14·9118	22·7509
10·8	0·000399	16·2414	24·4756
11·2	0·000264	17·6294	26·2663
11·6	0·000170	19·0754	28·1281
12·0	0·000105	20·5792	30·0592
12·4	0·000062	22·1405	32·0591
12·8	0·000034	23·7589	34·1276
13·2	0·000017	25·4344	36·2645
13·6	0·000007	27·1664	38·4695
14·0	0·000002	28·9549	40·7424
14·4	0·0000004	30·7996	43·0830
14·8	0·0000000	32·7003	45·4911
15·2	...	34·6567	47·9667

$\xi_1 = 14{\cdot}9715,$

$\psi_0(\xi_1) = 33{\cdot}5327, \quad \psi_2(\xi_1) = 46{\cdot}5444,$

$\psi_0'(\xi_1) = 4{\cdot}8812, \quad \psi_2'(\xi_1) = 6{\cdot}1766.$

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Copenhagen :
1933 *January* 4.

PHOTOGRAPHIC OBSERVATIONS OF COMETS MADE AT THE ROYAL OBSERVATORY, GREENWICH, 1926–27.

G. Merton, M.A., Ph.D.

(Communicated by the Astronomer Royal)

Through the kindness of the Astronomer Royal I was enabled to obtain some photographic observations of comets with the 30-inch reflector, at times when it was available for such work between 1927 February and August, and with short focus cameras mounted on the 13-inch astrographic telescope.

The work with the short focus cameras was started with a Ross portrait lens of 17-inch focal length working at $F/5\frac{1}{2}$ to see what could be done with such an instrument. This work was continued after 1926 November 13 with an Aldis aeroplane lens and a Cooke Aviar lens, both of 20-inch focal length and working at $F/5{\cdot}6$. They gave excellent definition over nearly the whole of a 5×4-inch plate. The results show that the accuracy of the deduced positions is surprisingly good, especially when, as in this case, duplicate exposures are made ; it is in fact of the same order as the best visual observations of comets, say $\pm 2''$. The results obtained with the 30-inch reflector are, of course, in general entitled to greater weight.

In Table I the last column indicates the instrument used, 30-inch reflector (R) or short focus cameras (S), and the number of plates or exposures on which each result depends. The cometary magnitudes given are based on careful estimates of the integrated light. One plate of Comet Pons-Winnecke taken by Mr. Witchell on June 23 with the 13-inch astrographic (A) was also reduced, and is included.

Table II gives the necessary information concerning the comparison stars used in the reductions.

TABLE I

Comet 1926 f—Comas Sola

	U.T. 1926	α 1926·0	δ 1926·0	$\log p_\alpha \Delta$	$\log p_\delta \Delta$	Comp. Stars	Mag.	Exp.
		h m s	° ′ ″					
1	Nov. 10·1394	2 51 35·43	+ 6 49 16·3	9·485	0·809	1 to 3	12	S3
2	,, 12·9296	2 48 46·63	7 0 6·7	9·014*n*	0·789	4 and 5	11½	S2
3	,, 14·9973	2 46 40·66	7 8 58·9	8·834	0·788	6 and 7	12	S2
4	Dec. 24·9133	2 21 41·59	12 24 58·9	9·218	0·750	8 and 9	11½	S2
	1927	α 1927·0	δ 1927·0					
5	Jan. 27·86425	2 44 26·29	+19 30 41·5	9·361	0·697	10 and 11	12·3	S2
6	Apr. 30·92346	6 26 59·63	33 10 27·5	9·640	0·772	12 and 13	...	R1
7	May 1·92211	6 30 1·34	33 9 56·2	9·640	0·772	14 to 17	...	R1
8	,, 8·88425	6 51 8·88	32 9 30·5	9·635	0·724	18 and 19	...	R1

Comet 1927c—Pons-Winnecke

	U.T. 1927	α 1927·0	δ 1927·0	log $p_\alpha\Delta$	log $p_\delta\Delta$	Comp. Stars	Mag.	Exp.
		h m s	° ′ ″					
9	Feb. 25·06804	14 4 25·54	+23 49 13·6	9·335*n*	0·644	20 to 23	15	R1
10	May 1·02561	14 58 6·26	48 34 50·7	8·408	9·6199	24 to 27	9½	R1
11	June 19·98314	18 6 57·39	48 10 51·0	9·0051*n*	9·7373	28 to 34	...	R1
12	,, 23·00985	19 16 36·21	37 38 10·9	9·0899*n*	0·3412	35 to 38	...	A1

Comet 1927d—Stearns

13	Mar. 27·04112	15 9 22·27	− 1 43 20·0	9·250*n*	0·845	39 and 40	11·0	S2
14	Apr. 22·95817	14 48 6·06	+ 8 26 37·8	9·227*n*	0·783	41 to 44	10	R1

Comet 1927e—Grigg-Skjellerup

15	May 1·87917	6 51 20·88	+10 34 49·8	9·550	0·808	45 and 46	...	R1
16	July 5·96768	17 19 38·76	39 36 27·5	8·973	0·267	47 and 48	12	R1
17	,, 19·95255	17 49 17·07	28 44 44·8	8·950	0·535	49 to 52	...	R1

Notes.—1. Diameter about ½′. 2. Images not good. 3. Images faint and difficult to measure in the machine used, but the two give very accordant results (< ½″). Diameter about ½′. 4. Patchy sky, poor images. 5. Images very faint. 6. Image rather poor. 8. Moon-fogged; comet image rather difficult to measure. 9. Diameter 15″. 10. Diameter of condensation 12″, not circular. 11. A particularly satisfactory observation; deserves fullest weight. 12. Comet trail about 50″ is too long and faint for satisfactory measurement. 13. Good images; comet diameter 15″ central condensation, coma 40″ to 60″ towards 200° about. 14. Good image; condensation diameter 10″ about. 15. Plate moon-fogged; comet image rather difficult (in α). 16. Good comet image; central condensation 3″ to 6″ mag. 14; elliptical coma to 8″ towards 330°. 17. Diameter of comet centre 12″.

Table II

Comparison Stars

	B.D.	Mag.	α 1926·0	δ 1926·0	Authorities
			h m s	° ′ ″	
1	+ 6° 447	9·2	2 51 32·64	+ 6 39 53·0	G.F.H.
2	Anon	10	2 51 39·98	6 52 33·3	Toulouse Astrographic.
3	6° 448	8·5	2 51 49·24	6 58 54·0	G.F.H.
4	6° 441	9·4	2 48 53·57	7 2 4·4	G.F.H.
5	6° 442	9·5	2 49 5·10	6 54 17·0	G.F.H.
6	6° 432	9·5	2 45 24·26	7 1 39·0	G.F.H.
7	6° 436	8·1	2 47 39·50	7 14 43·7	G.F.H.
8	11° 336	8·5	2 21 50·11	12 11 20·0	G.F.H.; 2nd Bordeaux Cat. +12° 86.
9	12° 332	7·8	2 22 26·20	12 33 57·7	G.F.H.; 2nd Bordeaux Cat. +12° 87.
			α 1927·0	δ 1927·0	
10	18° 347	7·7	2 43 19·88	+19 4 12·9	G.F.H.; Abbadia 1235 (Double star); Schorr PM.
11	19° 424	8·3	2 44 12·21	19 42 22·4	G.F.H.; Abbadia 1241.
12	32° 1316	7·9	6 26 13·09	32 51 22·9	A.G. (Leiden) 2649; Berlin Bab. 2432.
13	33° 1342	9·2	6 27 1·40	33 37 39·2	,, ,, 2659; ,, ,, 2438.
14	32° 1326	9·2	6 28 2·23	32 44 5·1	,, ,, 2675; ,, ,, 2447.
15	33° 1354	8·5	6 28 33·62	33 23 40·0	,, ,, 2681; ,, ,, 2451.
16	33° 1364	9·8	6 30 41·67	33 3 56·3	Bo VI (see G.F.H.); ,, ,, 2463.
17	33° 1370	8·5	6 32 13·28	33 41 58·1	G.F.H.; ,, ,, 2474.
18	33° 1428	8·2	6 49 27·47	33 5 37·2	G.F.H.; ,, ,, 2582.
19	33° 1438	9·2	6 53 13·26	32 58 24·9	G.F.H.; ,, ,, 2605.
20	24° 2685	8·4	14 2 55·94	+23 44 11·5	A.G. Berlin B 4971; Abbadia 7407.
21	23° 2651	8·9	14 3 51·83	23 29 43·9	,, ,, 4976; ,, 7415.
22	24° 2689	9·9	14 4 20·68	24 5 2·0	Abbadia 7419; Greenw. (1910) 6335.

	B.D.	Mag.	α 1927·0	δ 1927·0	Authorities
			h m s	° ′ ″	
23	+ 23° 2657	8·5	14 6 18·36	+23 29 57·6	A.G. Berlin B 4991 ; Abbadia 7438 ; Schorr PM.
24	48° 2252	7·9	14 55 39·46	+48 19 16·4	O-Arg.; A.G. Bonn (verb.) 9716 ; Greenw. (1925) MS.
25	49° 2344	8·9	14 56 59·50	48 58 7·2	A.G. Bonn (verb.) 9726 ; G.F.H. (MS.).
26	48° 2256	8·4	14 59 26·44	48 13 36·4	,, ,, ,, 9749 ; G.F.H. (MS.).
27	49° 2350	9·0	15 1 20·13	48 45 6·0	,, ,, ,, 9764 ; G.F.H. (MS.).
28	48° 2635	7·8	18 3 1·18	48 7 43·5	,, ,, ,, 11687 ; Groombr. 2510; Palermo 24·435 ; Greenw. (1925) MS.
29	48° 2638	7·7	18 4 25·20	48 50 18·1	A.G. Bonn (verb.) 11709 ; Greenw. (1925) MS.
30	48° 2639	7·9	18 4 34·14	48 35 12·1	A.G. Bonn (verb.) 11712 ; Palermo 24·439 ; Greenw. (1925) MS.
31	48° 2649	8·0	18 7 59·40	48 23 4·0	A.G. Bonn (verb.) 11763 ; Kü. 8048 ; Palermo 24·447 ; Bergedorf (1925) MS. ; A.J. 830 ; Greenw. (1925) MS.
32	48° 2651	8·0	18 9 12·81	48 0 38·1	A.G. Bonn (verb.) 11778 ; Bonn XVI 1437 ; Leiden MS. 729.
33	47° 2596	7·9	18 9 36·68	47 38 26·0	A.G. Bonn (verb.) 11785 ; Leiden MS. 731; Greenw. (1925) MS.
34	48° 2652	7·3	18 9 40·20	48 16 17·8	A.G. Bonn (verb.) 11786 ; Kü. 8059 ; A.J. 830 ; Rom-Camp (00) 2948 ; Palermo 24·449 ; Greenw. (1925) MS.
35	37° 3398	4·5	19 13 50·03	38 0 9·7	(θ Lyræ). Boss P.G.C. 4912 ; Greenw. (1925) MS.
36	37° 3411	8·8	19 16 26·09	+37 16 29·5	A.G. Lund 8309 ; Lund (1925) 8589.
37	38° 3538	8·2	19 16 50·63	38 23 39·6	,, ,, 8316 ; ,, ,, 8596.
38	37° 3432	6·7	19 20 6·79	37 26 22·8	Lund (1925) 8646 ; Greenw. (1925) MS.
39	− 1° 3030	7·3	15 7 48·19	− 1 59 4·5	Yale V 3881 ; Abbadia 8357.
40	− 1° 3036	8·6	15 10 15·04	− 1 36 51·7	,, 3890 ; ,, 8374.
41	+ 8° 2925	7·3	14 46 7·25	+ 8 17 19·3	Mün. 10614 ; A.G. Leipz. II 6818.
42	8° 2929	9·4	14 48 15·03	8 21 38·1	Toulouse IV 2289.
43	8° 2931	7·9	14 49 4·43	8 4 26·4	,, 2292 ; A.G. Leipz. II 6827.
44	9° 2953	8·4	14 49 6·84	8 46 46·5	Mün. 10665 ; A.G. Leipz. II 6829.
45	10° 1325	8·5	6 50 46·77	+10 38 38·0	G.F.H. (A.G. ; W. ; Lal.).
46	10° 1329	8·9	6 51 52·33	10 30 15·3	G.F.H. (A.G. ; Toulouse 1900 I, 1064).
47	39° 3104	7·8	17 17 17·04	39 21 48·9	A.G. Lund 7087 ; Greenw. (1925) MS.
48	39° 3120	8·8	17 21 33·07	39 48 33·0	,, 7115 ; Berlin Bab. 6370.
49	29° 3126	5·6	17 47 33·04	29 20 26·3	Boss P.G.C. 4510 ; Greenw. (1910) 7825 ; Pulkowa (1925) 937.
50	28° 2849	9·0	17 48 21·73	28 19 57·6	G.F.H. ; Greenw. (1910) 7835.
51	28° 2858	8·6	17 50 17·87	28 36 56·5	A.G. Camb. 8507 ; Greenw. (1910) 7862.
52	29° 3135	8·8	17 50 23·02	29 30 9·2	G.F.H. ; Greenw. (1910) 7864.

Notes.—Systematic corrections to the star places to bring them to Boss's system have been applied wherever known. Star 24, deduced PM $-^{s}\cdot011$, $-''\cdot06$. Star 31, revised PM $-^{s}\cdot003$, $+''\cdot075$. Star 34, revised PM (dec.) $-''\cdot051$.

ON PERIODIC OR SECULAR CHANGES OF VELOCITY ON JUPITER.

A. Stanley Williams.

(1) *The North Tropical Current*

Systematic observations of Jupiter may be said to have only commenced in the early 'eighties of last century, so far at least as the rotation periods or velocities of the chief surface currents are concerned. Before then we have only more or less isolated observations of a few individual spots, some of them, it is true, excellent observations and made by eminent and experienced observers. But at best the earlier record can only be described as a broken and very incomplete one.

We are now, however, fortunate enough to possess an accurate and nearly continuous record of the velocities of the chief surface currents of the planet covering about half a century. So that it appears possible at the present time to consider with some prospect of success the interesting subject of periodic or secular changes of velocity occurring in some of the principal surface currents of the planet. Some attempt is made here to do this in connection with the North Tropical Current.

This current is one of the best observed of all the currents, and one least affected by interference from prejudicial obstructions. In the corresponding region south of the equator we have the serious obstacles caused by the presence of the Red Spot and the South Tropical Disturbance, but there is nothing similar affecting the N. Tropical Current. The breadth of this current is roughly 10° of zenographical latitude, and it usually extends from about 10° N. lat. to 20° N. lat. It includes within its limits the usually very bright N. Tropical Zone and the northern half of the great N. Equatorial Belt, at least when the latter is broad or double. But when the last-mentioned belt is narrow or single it may be outside the current altogether, the southern limit of the latter falling just north of the northern edge of the belt. This occurred, for example, in 1898.* The actual limits of the current may shift somewhat in latitude from time to time, but the current itself is always there. It is a permanent feature of the planet.

Table I gives briefly the observed period of rotation of the N. Tropical Current in the different years. The material on which the values in the second column have been based and the system of weighting in the third column of the table will be explained in the latter part of this paper. The table is represented in diagram form in the adjoining figure (p. 411), which reveals several interesting features. The chief of these is the presence of very definite peaks in or about the years 1884, 1898, 1911, and 1922.

* See *Pub. A.S.P.*, **11**, 171, 1899. Judging from Mädler's well-known drawings of 1835 the entire belt, then very broad, was almost wholly included in the N. Tropical Current.

TABLE I

Observed Rotation Period of N. Tropical Current

Year	Rotation Period	Wt.	Year	Rotation Period	Wt.
	h m s			h m s	
1881	9 55 35·2	1	1906·7	9 55 26·2	3
1885	32·3	1	1907·8	20·0	3
1887	36·0	3	1908·9	25·0	3
1888	39·9	3	1911	16·4	2
1889	39·2	3	1914	32·0	3
1890	34·5	2	1915	27·5	3
1894·5	33·8	3	1916·17	30·1	3
1895·6	33·0	1	1917·18	28·1	3
1897	31·6	2	1918·19	27·7	3
1898	26·6	3	1919.20	30·9	3
1899	29·2	3	1920·21	28·5	1
1900	30·3	3	1922	20·6	3
1901	29·9	3	1923	23·5	2
1902·3	28·2	3	1926	32·1	3
1903·4	31·1	3	1927·8	33·5	3
1904·5	32·5	3	1929·30	28·1	2

A peak means that the current was then at its maximum velocity, and the presence of these four peaks indicates a distinct periodicity, the length of the period being about twelve years. The times of maximum velocity actually adopted were those given in Table II.

TABLE II

Observed and Computed Dates of Maximum Velocity

E	Maximum Velocity		O − C
	Observed	Calculated	
			y
−6	(1835)	1835·5	−0·5
−4	(1862)	1860·3	+1·7
−2	1884·5 ±	1885·1	−0·6
−1	1897·6	1897·5	−0·1
0	1910·3	1909·9	+0·4
+1	1922·0	1922·3	−0·3

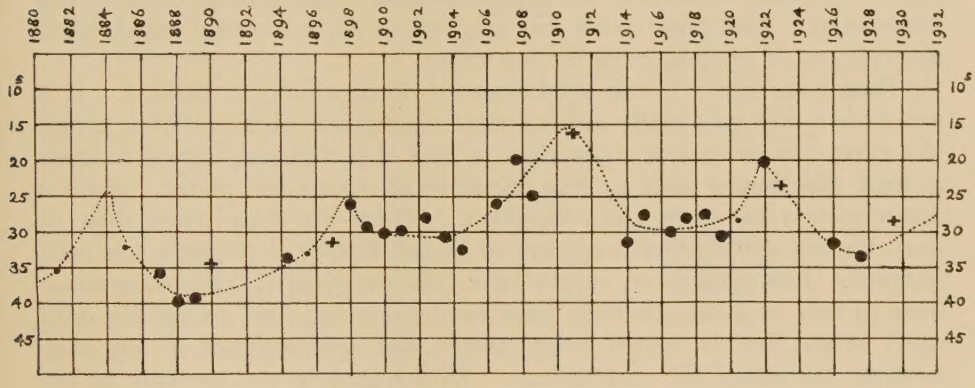
The residuals O − C have been computed from the following elements,

$$\text{Maximum Velocity} = 1909{\cdot}9 + 12^{y}{\cdot}4E$$

which satisfy with quite remarkable accuracy not only the four maxima above referred to, but also two additional observations included in Table II. These latter observations require some explanation. There are only a very few determinations of rotation period previous to 1880, and it is impossible to infer the exact date of any maximum from these. But there are two observations which were apparently made when the current was not far from its maximum velocity. In 1862 Julius Schmidt observed an isolated spot for a period of nearly two months, and derived a rotation period of

$9^h\ 55^m\ 25^s{\cdot}7$.* The computed date of maximum is 1860·3, so that the agreement seems good enough.

The other instance relates to the well-known pair of dark spots observed by Airy and Mädler in 1835. The latter derived a rotation period of $9^h\ 55^m\ 26^s{\cdot}5$ and the former one of $9^h\ 55^m\ 21^s{\cdot}3$. The method of observation employed at Cambridge † (the actual observer was Glaisher) was an unusual one, consisting in noting the times of transit of the spots and the limbs of the planet across the wires in the eyepiece of the equatorial. A rotation period of $9^h\ 55^m\ 24^s$ may be assumed from the two results, and as there were two spots and the observations extended over a period of about five months, this



Velocity Curve of North Tropical Current.

Vertical scale, $60^s = 72$ *km./h.* ● = *weight* 3, + = *weight* 2, • = *weight* 1.

period must be a reliable one. The velocity of the current must certainly have been above the average, and was probably not very far from its maximum. The computed date of maximum velocity is 1835·5.

The period 12·4 years is somewhat longer than that of the planet's revolution round the Sun (11·86 years), and is so much longer than the famous sunspot period that the one can hardly be related to the other. The extreme amplitude of the variation of rotation period is from about $9^h\ 55^m\ 15^s$ to $9^h\ 55^m\ 40^s$, that is a difference of 25^s. A difference of one minute in the length of the rotation period near the planet's equator is equivalent to a difference of about 72 kilometres (45 miles) per hour in the velocity of a current. So that the observed range of 25^s corresponds to about 30 km./h. in the actual velocity of the N. Tropical Current, not a very large change of velocity, it is true, but quite interesting.

The maximum of 1884 can only be roughly inferred because of the insufficient material. The value given for 1881 in Table I is that derived by O. Lohse from observations of a single spot ; ‡ that for 1885 from a series of observations by Denning,§ also of a single spot. And it is well known that

* *A.N.*, No. 1973, 1873.

† *Mem. R.A.S.*, 9, 1, 1836. At this time Airy was still at Cambridge.

‡ *Publications, Potsdam Observatory*, No. 9, p. 50, 1882.

§ *The Observatory*, 1885, p. 304.

a result from a single spot is not a reliable one, owing to the possibility of local or temporary variations or to real changes of appearance. In fact, previous to the writer's published observations of 1887 * there appear to be no available systematic observations of Jupiter suitable for investigations such as the present one.

It is necessary to explain the sources from which the rotation periods of Table I have been derived. Most of those previous in date to 1904 have been taken from the table published on p. 99 of vol. ii of the writer's *Zenographical Fragments*,† which table includes nearly all the reliable published determinations of rotation period. The subsequent data have been taken from the various *Memoirs* of the Jupiter Section of the British Astronomical Association. The final result for 1929–30 is derived from unpublished observations of 13 spots made by the writer.

Then with regard to the weights in the third column of Table I. It is well known that for a really satisfactory determination of the rotation period of a current in any year it is necessary to have observations of about 12 spots at least distributed more or less regularly all round the planet. Such a determination has been given weight 3. Where there are fewer than 12 spots but the determination appears to be satisfactory, a weight 2 has been assigned. Less satisfactory results have been weighted 1. The determinations of two or more observers have been combined, not necessarily with equal weight, but no higher weight than 3 has been assigned in any case, however numerous the observations. With regard to the accuracy of the results in the table, it is safe to assume that one weighted 3 is not likely to be in error by as much as one second of time.

In conclusion the writer would like to acknowledge his great indebtedness to the various *Memoirs* published by the Jupiter Section of the British Astronomical Association. These *Memoirs* constitute a vast storehouse of interesting facts and valuable data relating to the planet such as is accessible nowhere else.

Summary

(1) The systematic observations of the last half century indicate a definite periodicity in the velocity of the N. Tropical Current of Jupiter with a period of slightly over twelve years.

(2) The amplitude or range of the variation in velocity amounts to 30 km./h.

(3) The rotation period of the current varies from $9^h\ 55^m\ 15^s$ to $9^h\ 55^m\ 40^s$.

(4) The concluded elements of variation are

$$\text{Maximum Velocity} = 1909{\cdot}9 + 12^y{\cdot}4E.$$

(5) The computed date of the next epoch of maximum velocity is 1934·7.

* *Zenographical Fragments*, 1 (London, 1889).

† *Ibid.*, 2 (London, 1909).

Polvarth, St. Mawes:
1933 *February* 11.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE NATIONAL OBSERVATORY OF ATHENS.

S. Plakidis and G. Adamopoulos.

Ref. No.	Star	Mag.	Date 1932	Local Sider. Time h m s	Observer
1	B.D. +6° 2369	6·5	Aug. 4	16 58 3·8	Ad.
2	,, +0° 2833	8·8	5	16 57 42·2	,,
3	Cordoba B 10249	8·4	10	18 28 54·4	,,
4	,, ,, 10273	8·3	10	19 21 58·2	,,
5	,, ,, 10274	8·5	10	19 24 8·3	,,
6	,, ,, 11013	8·9	11	17 55 44·9	Pl.
7	,, ,, 11031	9·0	11	18 42 37·9	Ad.
8	,, ,, 11039	9·2	11	19 8 42·0	,,
9	,, ,, 11047	8·9	11	19 25 12·2	,,
10	,, ,, 11043	7·5	11	19 35 57·9	,,
11	,, ,, 11055	9·0	11	19 40 15·9	,,
12	,, ,, 11992	8·2	12	17 32 48·1	,,
13	,, ,, 11993	9·0	12	17 38 57·0	,,
14	,, ,, 12026	7·2	12	19 2 3·9	Pl.
15	,, ,, 12044	8·6	12	19 54 7·9	,,
16	,, ,, 12756	6·2	13	16 52 28·8	,,
17	,, ,, 12820	8·4	13	19 5 20·9	Ad.
18	,, ,, 12821	9·0	13	19 11 47·0	,,
19	B.D. −22° 3886	8·5	Sept. 5	18 0 50·8	,,
20	,, −22° 3888	9·0	5	18 20 6·9	,,
21	Cordoba B 11726	7·9	8	18 23 45·4	,,
22	,, ,, 11732	8·8	8	18 39 44·0	,,
23	,, ,, 11738	8·0	8	18 42 25·6	,,
24	,, ,, 11743	9·0	8	18 49 47·7	,,
25	,, ,, 11745	8·8	8	18 52 52·8	,,
26	,, ,, 11759	9·2	8	19 22 40·6	Pl.
27	,, ,, 11760	9·1	8	19 23 24·0	,,
28	,, ,, 11770	9·2	8	19 45 49·6	,,
29	,, ,, 11769	9·0	8	19 52 12·8	,,
30	,, ,, 11779	9·1	8	20 17 59·6	,,
31	,, ,, 11777	9·3	8	20 18 34·9	,,
32	,, ,, 11798	8·6	8	20 53 16·1	,,
33	,, ,, 11803	9·0	8	21 4 51·4	,,
34	,, A 14042	8·4	10	18 41 18·9	,,
35	,, ,, 14047	8·6	10	18 49 33·4	,, *
36	,, ,, 14061	8·6	10	20 33 54·1	Ad.
37	,, ,, 14088	8·6	10	22 6 25·4	,,
38	,, ,, 14090	8·6	10	22 10 54·3	,,
39	B.D. −3° 5657	9·0	14	21 18 10·3	,,
40	,, −3° 5655	6·8	14	21 31 18·5	,,
41	,, −4° 5907	9·2	14	21 45 38·3	,,
42	Cordoba B 10526	8·0	Oct. 4	19 31 25·0	,,

* Hazy, uncertain.

Ref. No.	Star	Mag.	Date 1932	Local Sider. Time h m s	Observer
43	Cordoba B 11407	8·8	Oct. 5	19 47 50·2	Ad.
44	,, ,, 11417	8·8	5	20 22 47·9	,,
45	B.D. −5° 5966	6·2	11	22 6 34·5	Pl.
46	,, −5° 5972	8·6	11	23 41 9·3	,,
47	,, −5° 5973	7·0	11	23 56 51·3	,, *
48	,, +5° 104	6·8	13	22 59 27·3	,,
49	,, +6° 105	7·0	13	0 12 11·0	,,
50	,, +6° 107	4·5	13	1 14 52·0	,,
51	Cordoba B 12058	9·0	Nov. 2	20 40 16·2	Ad.
52	,, ,, 12062	8·7	2	20 51 2·6	,,
53	,, ,, 12070	8·8	2	21 20 1·2	,,
54	,, ,, 12080	7·0	2	21 59 7·5	,,
55	,, ,, 12090	8·7	2	22 8 30·9	,,
56	,, A 13674	9·0	3	21 8 41·6	,,
57	,, ,, 13673	9·2	3	21 16 38·5	,,
58	,, ,, 13679	9·1	3	21 19 32·9	,,
59	B.D. −7° 5926	9·3	7	22 4 25·6	,,
60	,, +9° 146	7·8	10	3 5 38·4	,,
61	,, +13° 275	8·9	Dec. 8	0 19 11·6	,,

* Uncertain.

Instrument used :—Doridis refractor (Gautier, 400 mm.).
Magnifying power :—155 times.
Phase observed :—Disappearances at the dark limb.
Observers :—S. Plakidis, D.Sc. (Pl.), and G. Adamopoulos (Ad.).

Erratum.

L. J. Comrie, *The Computation of Total Solar Eclipses*, p. 178, tenth equation from top, *for* ζ *read* ζ_1.